



Georgia Southern University
Digital Commons@Georgia Southern

Electronic Theses and Dissertations

Graduate Studies, Jack N. Averitt College of

Spring 2016

Improving The Inference Of Some Experiemtal Studies By Using Ranked Auxiliary Covariates

Rajai Jabrah

Follow this and additional works at: <https://digitalcommons.georgiasouthern.edu/etd>

 Part of the [Behavior and Behavior Mechanisms Commons](#), [Clinical Epidemiology Commons](#), and the [Epidemiology Commons](#)

Recommended Citation

Jabrah, Rajai, "Improving The Inference Of Some Experiemtal Studies By Using Ranked Auxiliary Covariates" (2016). *Electronic Theses and Dissertations*. 1404.
<https://digitalcommons.georgiasouthern.edu/etd/1404>

This dissertation (open access) is brought to you for free and open access by the Graduate Studies, Jack N. Averitt College of at Digital Commons@Georgia Southern. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of Digital Commons@Georgia Southern. For more information, please contact digitalcommons@georgiasouthern.edu.

IMPROVING THE INFERENCE OF SOME EXPERIMENTAL STUDIES BY USING
RANKED AUXILIARY COVARIATES

by

RAJAI JABRAH

(Under the Direction of Hani M. Samawi)

ABSTRACT

In many studies, the measurement of sampling units according to the response variable is costly or time consuming, however, it is possible to rank sampling units according to baseline auxiliary covariates, which are available, easily obtainable, and cost efficient. In these cases, when estimating the population mean, *Ranked Set Sampling* (RSS) can be a more efficient sampling method than the *Simple Random Sampling* (SRS) method. In this dissertation, we propose a modified approach of the RSS method to allocate units into an experimental study, aimed to compare L groups.

Ranked auxiliary covariates, which are typically correlated with the variable of interest, are involved in sampling design; these covariates are available and affordable. Computer simulation is used to estimate the empirical nominal values and the empirical power values for the modified RSS, by using the regression approach in analysis of covariance (ANCOVA) models, and compared to the SRS. Results indicate that the required sample sizes for a given precision are smaller under RSS than under SRS.

The modified RSS protocol was applied to an experimental study conducted by the Department of Psychology, in collaboration with the College of Public Health, Department of Biostatistics, at Georgia Southern University. The experimental study was designed to obtain a better understanding of the pathways by which positive experiences (i.e., goal completion) contribute

to higher levels of happiness, well-being, and life satisfaction. Using the RSS method resulted in significant cost reduction associated with smaller sample size without losing the significant precision of the analysis.

INDEX WORDS: Ranked Set Sampling (RSS), Ranked Auxiliary Covariate, Sampling Design, Cost-effectiveness, Reduced Sample Size, Powerful, Emotional Uplifting Intervention, ANCOVA, Multiple Regression

IMPROVING THE INFERENCE OF SOME EXPERIMENTAL STUDIES BY USING
RANKED AUXILIARY COVARIATES

by

RAJAI JABRAH

B. S., Applied Science University, Jordan, 2006

MPH., Georgia Southern University, 2009

A Dissertation Submitted to the Graduate Faculty of Georgia Southern University in

Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PUBLIC HEALTH

STATESBORO, GEORGIA

© 2016

RAJAI JABRAH

All Rights Reserved

IMPROVING THE INFERENCE OF SOME EXPERIMENTAL STUDIES BY USING
RANKED AUXILIARY COVARIATES

by

RAJAI JABRAH

| | |
|------------------|------------------|
| Major Professor: | Hani M. Samawi |
| Committee: | Robert L. Vogel |
| | Daniel F. Linder |
| | Haresh Rochani |

Electronic Version Approved:
May 2016

ACKNOWLEDGMENTS

A multitude of thanks to my advisor, Dr. Hani Samawi, for his continual guidance and encouragement throughout this dissertation. His unceasing patience and diligence has helped me move forward, step-by-step, during difficult situations. Besides being one of leading researchers in the field of Biostatistics, he is also one of the most respectable people one could ever associate with. Without his great guidance and patience, it would have been much more difficult for me to complete the Doctorate degree. His efforts will always be remembered, and his support will always be recognized as the best help I have ever received at Georgia Southern University.

Special thanks go to the committee members; I could never have gotten this far without their help and support. Dr. Robert Vogel who has always welcomed me when I would drop by, and who has helped me with several key ideas. Furthermore, Dr. Daniel Linder and Dr. Haresh Rochani, both have been the greatest advisors a student could hope for. I would like to thank them for their advice, and I greatly appreciated their input. In addition, I would like to thank Dr. Karl Peace for his useful and valuable advises in clinical research.

I would like to thank Dr. Jeff Klibert from the Department of Psychology for his collaboration with the experimental study. I also thank the other professors from the Jiann-Ping Hsu College of Public Health, who through their teaching and patience during my graduate years have enabled me to achieve this goal.

Finally and foremost, I would like to thank my father, my mother, and my wife for their unceasing love and support throughout my life. I would not be able to finish my dissertation without them and their continual support and encouragement.

TABLE OF CONTENTS

| | |
|---|----|
| ACKNOWLEDGMENTS..... | 2 |
| LIST OF TABLES | 5 |
| LIST OF FIGURES | 6 |
| 1. INTRODUCTION | 7 |
| 1.1. Background | 7 |
| 1.2. The Use of Auxiliary Covariates | 8 |
| 1.3. Sampling Designs: Simple Random Sampling and Ranked Set Sampling | 9 |
| 2. LITERATURE REVIEW | 11 |
| 3. METHOD | 16 |
| 3.1. Objective | 16 |
| 3.2. Analysis Of Covariance Approach | 16 |
| 3.3. Parameters Estimation: Least-Squares Estimator | 19 |
| 3.4. Sampling Method: Ranked Set Sampling | 19 |
| 4. ON THE INFERENCE OF COVARIANCE ANALYSIS USING RANKED AUXILIARY COVARIATES | 24 |
| 4.1. Introduction | 24 |
| 4.2 The Modified Ranked Set Sampling Method | 24 |
| 4.3. ANCOVA Models To Compare Groups | 26 |
| 4.4. Some Asymptotic Results of Using the Modified Ranked Set Sampling | 29 |
| 4.5. Estimating Unknown Parameters: Weighted Least Squares | 33 |
| 4.6. Hypothesis Testing | 46 |
| 5. A SIMULATION STUDY | 47 |
| 5.1 Design of the Simulation Stud | 47 |
| 5.2 Results of Simulation | 48 |
| 5.3 Final Remarks of the Results | 63 |
| 6. BOOSTING HAPPINESS AND BUTTRESSING RESILIENCE: UPLIFTING INTERVENTION DESIGNED BY USING RSS APPROACH | 64 |
| 6.1. Objectives of the Chapter | 64 |
| 6.2. Introduction | 65 |
| 6.3. The Purpose of the Study | 66 |
| 6.4. Description of the Subjects | 66 |
| 6.5. Recruitment | 67 |
| 6.6. Intervention Groups | 69 |

| | |
|---|----|
| 6.7. Data Analysis | 70 |
| 7. CONCLUSION AND FUTURE WORK | 76 |
| 7.1. Summary | 76 |
| 7.2. Future Work | 78 |
| REFERENCES | 80 |
| APPENDICES | 85 |
| APPENDIX A: EMPIRICAL POWER PLOTS | 85 |

LIST OF TABLES

| | |
|--|----|
| Table 1: Empirical Nominal Value ($\alpha = 0.05$) For Two Interventional Groups | 49 |
| Table 2: Empirical Power For Testing Two Interventional Groups With Mean Different (d=0.3) | 51 |
| Table 3: Empirical Power For Testing Two Interventional Groups With Mean Different (d=0.5) | 52 |
| Table 4: Empirical Power For Testing Two Interventional Groups With Mean Different (d=0.8) | 53 |
| Table 5: Empirical Nominal Value ($\alpha = 0.05$) For Three Interventional Groups | 55 |
| Table 6: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.1 & d2=0.3) | 57 |
| Table 7: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.5 & d2=0.3) | 58 |
| Table 8: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.8 & d2=0.3) | 59 |
| Table 9: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.3 & d2=1.0) | 60 |
| Table 10: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.5 & d2=1.0)..... | 61 |
| Table 11: Empirical Power For Testing Three Interventional Groups With Mean Different (d1=0.8 & d2=1.0) | 62 |
| Table 12: Descriptive Characteristics By Intervention Groups: Uplifting Experimental Study .. | 72 |
| Table 13: Analysis of Covariance in a Multiple Regression Model | 74 |

LIST OF FIGURES

| | |
|---|----|
| Figure 1: Power Analysis For Positive Mood Outcome For Uplifting Intervention By Using RSS Scheme | 73 |
| Table 2 Figure 2: Uplifting Intervention and Control Final Positive Mood by Time | 75 |

Chapter 1

INTRODUCTION

1.1. Background

A major objective of conducting research is to make inferences about a larger population from a sample, which is representative of that population. Lack of sufficient funding and time are often challenges for researchers aimed at drawing accurate conclusions about the population under study. Therefore, sampling design is one of the critical elements of the research protocol used to assist in the selection of a true representative sample of the population (Sandelowski, 2000). Probability sampling techniques are used to obtain unbiased results and to minimize sampling errors, where each subject of the population has a known, non-zero chance of being selected into the sample (Kandola et al., 2014).

In some areas such as: ecological, environmental, agricultural, and epidemiological studies, there are situations where the measurement of the response variable is costly to obtain or time consuming. However, other information, such as an auxiliary covariate, is correlated with the variable of interest, and can be easily obtained at a lower cost. The auxiliary covariates can be used to select a smaller sample and reduce the cost of the study (Chen et al., 2004). Therefore, cost efficient sampling methods are desirable in order to reduce cost and increase efficiency.

In this dissertation, we will investigate two methodologies of obtaining data via sampling to compare two groups or more. One of those methodologies is the *Simple Random Sampling* (SRS). The other methodology is called *Ranked Set Sampling* (RSS). RSS was proposed by McIntyre (1952) to be used in environmental and ecological studies as an alternative method to SRS. RSS was designed to estimate the mean with improved efficiency. When adopting RSS, it

is assumed that a sample from a population can be inexpensively and easily ordered by direct ranking methods such as visual inspection, or by indirect ranking methods based on an auxiliary covariate that is highly correlated with the variable of interest.

1.2. The Use of Auxiliary Covariates for Comparing Groups

Along with the experimental outcome of interest, other information such as demographic and physiological characteristics, which include age, gender, weight, blood pressure measures, etc. might be recorded. These are called "auxiliary covariates." Some of these auxiliary covariates can be highly correlated to the outcome of interest. Therefore, the information contained in the covariates can be used for several purposes, including subject selection and for improving inference efficiency when comparing L groups (Chen et al., 2008).

For example, determining the age of a fish involves taking one of the otoliths from the fish, embedding the cleaned otolith onto a microscope slide, and finally counting the daily rings of the otolith under immersion oil; this process is time consuming and expensive (Chen & Shen, 2003; Soekoe et al., 2013). However, other information can be collected and ranked easily such as the length and weight of the fish. Both length and weight are correlated with age. In this dissertation, we propose to use an efficient sampling procedure for group comparison using a modified technique of RSS. Other researchers such as: Halls and Dell (1966), Evans (1967), Stokes and Sager (1988), and Takahasi and Wakimoto (1968) have shown the effectiveness of RSS over SRS. The key here is that indirect ranking by ordering on some auxiliary covariate (X), which are highly correlated with the variable of interest (Y) can lead to more precise estimation of population parameters.

Many researchers used the RSS technique in regression analysis. For example, Sinha et al. (1993) proposed regression estimators with ranking on the auxiliary covariates. They demonstrated that the estimator based on RSS is always more efficient than the regression estimator based on SRS. However, to assess the relationship between the variable of interest (Y) and risk factors and/or covariates (X) in different groups, complex models such as multiple linear regression and analysis of variance (ANOVA) have received very little attention, see Muttalak (1996). This dissertation proposes ranked-based estimation and testing procedures for comparing the means of the group's response variable using the analysis of covariance (ANCOVA).

1.3. Sampling Designs: Simple Random Sampling and Ranked Set Sampling

Simple Random Sampling (SRS) is the most basic form of probability sampling and provides the theoretical basis for more structured forms of sampling such as RSS. In SRS a sample is drawn from a population of size N in such a way that every possible sample of size n has the same chance of being selected (Daniel, 2012). Some advantages of using the SRS method are that no restrictions are placed on the nature of selection, and the sample size is easy to obtain, determine, and is straightforward (Dura et al., 2010). Another important advantage of the SRS is that statistical procedures required to analyze data are easier to perform than those required for other probability sampling procedures, such as in Stratified Random Sampling and Cluster Random Sampling (Daniel, 2012). One of the disadvantages of this sampling technique is that it could lead to a non-representative sample if a small sample size was drawn from a large population. SRS tends to have larger sampling errors and less precision for a fixed sample size compared to other methods with the same sample size. On the other hand, to select a sample by using SRS, a complete list of the population members is required (Daniel, 2012).

However, RSS which was first introduced by McIntyre (1952) is an efficient alternative to SRS. The RSS is a two-stage sampling plan. At the first stage, other information than the measurements of the variable of interest are used to rank a set of sampling units, then, at the second stage, only one unit in the set with a specific rank is selected for the measurement of the variable of interest (Chen & Shen, 2003).

Consequently, the idea proposed in this project is to improve inference when comparing groups by incorporating ranked auxiliary covariates that are correlated to the variable of interest. To use this method effectively, we have the additional assumption that the response variable is difficult and expensive to obtain. In this study, a random sampling process is applied to large cohorts by using the RSS technique based on inexpensive auxiliary covariates, correlated with the outcome of interest. Our goal is to provide a more precise estimator of the population mean of the outcome of interest (Y), without making any additional assumptions other than those that are already necessary for RSS and the Weighted Ordinary Least Square estimators for ANCOVA models by using multiple regression approach.

Chapter 2

LITERATURE REVIEW

Many environmental, ecological, and medical studies are designed to assess the relations between exposures, or other factor variables (X), and the corresponding outcome variable of interest (Y). The primary goal of these studies is to provide a comparison of groups with maximum precision and validity (McEntegart, 2003). However, high cost of sampling design with large number of units is a major issue that may cause a limitation in precision (English et al., 2010). Therefore, there is an increasing attention in the literature to improve a study's efficiency while minimizing the cost. This effort is being accomplished by developing new statistical and sampling approaches to minimize the required sample size. Using inexpensive auxiliary covariates is one of the suggested strategies (Hauck et al., 1998; Koch et al., 1998; Lesaffre & Senn, 2003; Pocock et al., 2002; Senn, 1989; Tsiatis et al., 2008).

Egger et al. (1985) suggested that hypothesis testing could be improved by incorporating baseline information into the model under study. Donner and Zou (2007) showed that the efficiency of treatment comparisons could be improved if highly correlated baseline measurements and outcome measurements were accounted for in the statistical analysis. Tsiatis et al. (2008) proposed an approach of adjustment for auxiliary covariates to improve the inference of randomized clinical trials, called minimization. Zhang et al. (2008) also proposed an approach to adjust for auxiliary covariates to improve the precision of estimating treatment effects and the general null hypothesis in the analysis of randomized clinical trials by using semi-parametric theory. All the researchers showed that the use of auxiliary covariates, which are related to the outcome of interest, could improve efficiency. The use of baseline auxiliary

information is recommended in literature for reducing the required sample size that is needed in clinical studies (Egger et al., 1985).

To ensure reliable and valid inferences from a sample, a probability sampling technique is used to obtain unbiased results. Different sampling techniques have been proposed to ensure that the sample group is a true representative of the population without errors (Kalsbeek & Heiss, 2000). In practice, SRS might be preferred for some studies, since it is the simplest method to select subjects (Lachin, 1988). However, other sampling methods such as RSS can reduce the cost of sampling, and increase the precision of the inference, in certain situations where the measurement of the outcome of interest is costly (Lam et al., 2002). Dell and Clutter (1972) provided the mathematical foundations for RSS. They proved that the sample mean of the RSS was an unbiased estimator of the population mean with smaller variance than the sample mean of SRS with the same effective sample size.

Stokes (1977) was the first to consider the case where the ranking is done on the basis of an auxiliary covariate (X) instead of judgment for the response variable (Y). He proposed an estimator of the population variance based on RSS, and he showed that the estimator is asymptotically unbiased and more efficient than the sample variance of SRS, with the same number of observations quantified (Stokes, 1980). MacEachern et al. (2003) derived an unbiased estimator of the population variance based on RSS, and showed that the estimator was more efficient than its counterpart based on SRS, and more efficient than Stokes' estimator. Samawi and Pararai (2010) exposed that the RSS is possible and recommended as an alternative to SRS and it gives more precise results in certain epidemiological studies.

Since the RSS technique is considered an efficient alternative to the SRS method in agricultural and environmental studies, this method can also be used as an efficient way of

incorporating auxiliary information at the design stage of human studies (Samawi & Vogel, 2015). RSS has gained extensive attention and application in recent years by many researchers in clinical studies, by using available data from medical records, which are highly related to the medical measure under consideration, or by using some inexpensive medical screening tests. RSS can be used by ranking the selected random sample according to the data in those medical records or according to the result of that screening test. On the other hand, some medical conditions can be screened by visual inspection, and then the patients can be ranked according to their conditions. For example, Samawi and Al-Sagheer (2001) proposed a RSS to evaluate the level of bilirubin in the blood of infants. To establish normal ranges for the level of bilirubin in the blood of the jaundiced, premature babies, ranking the level of bilirubin in the blood can be done visually, for little cost, by observing the color of the face, chest, and terminal parts of the whole body. As the yellowing goes from the face to the terminal parts, the level of bilirubin in the blood increases. Chen et al. (2008) has investigated the use of various RSS protocols for the assignment of experimental units in treatment groups and investigated their properties. The authors have applied their proposed sampling scheme in an already developed clinical trial study aimed to evaluate patients with HIV-1 infection. They found that the proposed RSS method was more efficient than SRS by reducing the variance of the difference between the treatment sample means.

Parameters of the auxiliary covariates such as the mean, median, variance and the coefficient of variation can be used to increase the efficiency of the estimators. Samawi and Muttalak (1996) investigated RSS for estimating the population mean of the variable of interest (Y) using a ratio estimator, and they suggested that the RSS estimator of the population ratio was always more efficient than the SRS. Similar to ratio estimators, regression estimators are used to

estimate the population mean of the variable of interest based on RSS. Stokes (1977) first introduced the use of RSS in the linear regression model. He showed that RSS would have been a better choice than SRS for estimating the population mean by modeling the relation between auxiliary variables and the variable of interest for the linear regression model. In 1993, Sinha et al. published a work on RSS regression analysis. They indicated that the regression analysis based on perfect ranking of the RSS estimator is considerably more efficient than the SRS, especially when the correlation between the outcome and the auxiliary covariate is quite low. In addition, research has shown that ranked sample obtained by using RSS provides more efficient parameter estimation in regression model than those obtained by using SRS (Samawi & Abu-Dayyeh, 2002).

Muttalak (1996) considered RSS for simple and multiple regression models. For both cases, least squares estimation (LSE) was adopted for parameter estimation, and variances for the regression errors were assumed unequal. Barreto and Barnett (1999) considered a simple linear regression model with replicated observations obtained from RSS. They considered a case where the dependent variable was normally distributed and ranking was assumed to be perfect. It was remarked that the estimators of slope and intercept parameters based on RSS were more efficient than the estimators based on SRS. Samawi and Ababneh (2001), used RSS, ranking only on the variable (X), to investigate the effect of RSS on regression analysis in general. They found that it was more efficient to use RSS for regression model parameter estimation than SRS. However, Samawi and Abu-Dayyeh (2002) revealed that using *Extreme Ranked Set Sampling* (ERSS) improved the performance of estimating the regression model parameters. Furthermore, Özdemir and Esin (2007) examined the RSS method in the simple and the multiple linear regression model. They concluded that RSS provided a more efficient way to conduct a regression analysis.

In the next chapter, we will focus on the use of RSS in more complex models to select sampling units and their assignment to L groups. The mean of the response variable for the groups will be estimated in analysis of covariance (ANCOVA) models.

Chapter 3

METHODS

3.1. Objective

This study proposes a parametric method for inference by using a multiple regression approach in analysis of covariance model to compare L groups experimental conditions. In this chapter, we will explore the properties of the ANCOVA analysis under the modified RSS design. Assuming large cohorts of subjects are available to select from, ranking will be performed by estimating the baseline measures of a continuous auxiliary covariate.

3.2. Analysis of Covariance Models

The analysis of covariance (generally known as ANCOVA) is a technique that incorporates both analysis of variance (ANOVA) and regression analysis. Analysis of covariance is used when examining the differences of the mean values of the outcome between different groups that are related to the effect of the controlled independent variables. ANCOVA models explain the dependent variable by combining categorical, or indicator variables (also, often called “treatments”), and continuous independent variables (X). ANCOVA models can be fitted using multiple regression analysis. In this approach, ANCOVA is considered as a special case of the general linear model (GLM) framework. While the comparison is interested in the effects estimation of the specific factors chosen (treatments), the factor levels are considered fixed.

The two most important purposes of ANCOVA models, discussed by Culpepper and Aguinis (2011), are: 1) to increase the precision of comparisons between groups by accounting for variation in important auxiliary variables, and 2) to "adjust" comparisons between groups for

imbalances in important auxiliary variables between these groups. On the other hand, the choice of auxiliary covariates is an important step. If a covariate has no relation to the response variable, nothing is gained by covariance analysis. Therefore, a highly correlated auxiliary covariate with a response variable will be selected for the analysis. The model is written as,

$$y_{ij} = \mu + \gamma_i + \beta x_{ij} + \varepsilon_{ij} \quad \begin{array}{l} i = 1, 2, \dots, L \text{ groups} \\ j = 1, 2, \dots, n \text{ units} \end{array} \quad (3.1)$$

where y_{ij} is the response for the j^{th} subject in the i^{th} group, μ is the overall mean, γ_i is the usual group effect, ε_{ij} is the random error and x_{ij} is the auxiliary covariate. ANCOVA makes the following assumptions: ε_{ij} are identically and independently normally distributed; the slope β , is equal across different groups, such as treatment and control groups. The relationship between y_{ij} and x_{ij} is a linear condition, and homogeneity of variance is satisfied across the groups. Another important assumption of ANCOVA is that covariates are measured without error (Klockars & Beretvas, 2001; Linn & Werts, 1971).

In matrix notation, the model (3.1) can be written,

$$Y = \beta X + \gamma Z + \varepsilon, \quad (3.2)$$

where

- 1) Y is an $(nL \times 1)$ vector of the subjects' response variable;
- 2) X is an $(nL \times p)$ matrix of subjects' auxiliary covariates;
- 3) Z is an $(nL \times L-1)$ matrix of indicator variable for the groups; and
- 4) ε is an $(nL \times 1)$ vector of errors, given that $\varepsilon = Y - E[Y]$, with $E[e] = 0$ and $\text{var}[e] = \sigma^2 I$.

In case of one covariate (X) and L groups, we have:

$$Y = \begin{bmatrix} y_{1,1} \\ y_{1,2} \\ \vdots \\ y_{1,n} \\ y_{2,1} \\ y_{2,2} \\ \vdots \\ y_{2,n} \\ \vdots \\ y_{L,1} \\ y_{L,2} \\ \vdots \\ y_{L,n} \end{bmatrix}_{nL \times 1}, \quad X = \begin{bmatrix} 1 & x_{1,1} \\ 1 & x_{1,2} \\ \vdots & \vdots \\ 1 & x_{1,n} \\ 1 & x_{2,1} \\ 1 & x_{2,2} \\ \vdots & \vdots \\ 1 & x_{2,n} \\ \vdots & \vdots \\ 1 & x_{L,1} \\ 1 & x_{L,2} \\ \vdots & \vdots \\ 1 & x_{L,n} \end{bmatrix}_{nL \times 2}, \text{ and}$$

$$Z = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}_{nL \times (L-1)}$$

3.3. Parameters Estimation: Least-Squares Estimator

Some of the most common estimation techniques for ANCOVA models are Least-Squares Estimation (LSE), Maximum-Likelihood Estimation (MLE), and other estimation techniques such as Bayesian linear regression and principle component regression. In this dissertation, the unknown parameters in ANCOVA models are estimated by using the LSE technique.

The LSE is widely used to find, or estimate, the unknown values of the parameters to fit a function to a set of data and to characterize the statistical properties of estimates. The LSE method concerns estimating parameters by minimizing the sum of square errors. LSE exists with several variations; the simplest version is called Ordinary Least Squares (OLS), and a more sophisticated version is called weighted least squares (WLS).

To test whether or not the experimental groups differ in effectiveness in the fixed effects model, we can follow the general linear test approach of fitting full and reduced models. In this case, the alternatives are,

$$H_0 : \gamma_1 = \gamma_2 = \dots = \gamma_L = 0$$

$$H_A = \text{at least one group effect not equal zero}$$

3.4. Sampling Method: Ranked Set Sampling

The original form of RSS introduced by McIntyre (1952) is described as follows. First, a simple random sample of size (m) is drawn from the population and the (m) sampling units are ranked with respect to the variable of interest, say (Y) , or any other auxiliary covariates correlated with the variable of interest, say (X) . The unit with the first rank is identified and taken for the measurement; the remaining units of the sample are discarded. Next, another simple

random sample of size (m) is drawn and the units of the sample are ranked by judgment. The unit with the second rank is taken for the measurement, and the remaining units are discarded. This process is continued until the m^{th} simple random sample of size (m) is taken and ranked, and the unit with the m^{th} rank is taken for the measurement. This entire process refers to a single cycle. The cycle then repeats (r) times to yield a balanced ranked set sample of size $n = mr$. For example, for $m=3$, the sampling procedure can be illustrated as follows,

$$\begin{array}{c}
 \text{Cycle 1} \\
 Y_{[1]11} \leq Y_{[2]11} \leq Y_{[3]11} \Rightarrow Y_{[1]1} \\
 Y_{[1]21} \leq Y_{[2]21} \leq Y_{[3]21} \Rightarrow Y_{[2]1} \\
 Y_{[1]31} \leq Y_{[2]31} \leq Y_{[3]31} \Rightarrow Y_{[3]1} \\
 \\
 \text{Cycle 2} \\
 Y_{[1]12} \leq Y_{[2]12} \leq Y_{[3]12} \Rightarrow Y_{[1]2} \\
 Y_{[1]22} \leq Y_{[2]22} \leq Y_{[3]22} \Rightarrow Y_{[2]2} \\
 Y_{[1]32} \leq Y_{[2]32} \leq Y_{[3]32} \Rightarrow Y_{[3]2} \quad . \\
 \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 \\
 \text{Cycle } r \\
 Y_{[1]1r} \leq Y_{[2]1r} \leq Y_{[3]1r} \Rightarrow Y_{[1]r} \\
 Y_{[1]2r} \leq Y_{[2]2r} \leq Y_{[3]2r} \Rightarrow Y_{[2]r} \\
 Y_{[1]3r} \leq Y_{[2]3r} \leq Y_{[3]3r} \Rightarrow Y_{[3]r}
 \end{array}$$

In general, a total of $m^2 r$ units are sampled from the population of interest but only mr units are quantified to constitute a ranked set sample of size $n = mr$.

Furthermore, let S_j be the number of measurements of units in subset S with rank j ,

$j = 1, \dots, m$, such that $\sum_{j=1}^m S_j = n$. Let $Y_{[m]}$ denote a measurement of the m^{th} unit from the rank j , (j^{th} judgment order statistics).

Then the ranked set sample of size $n = \sum_{j=1}^m S_j$ with set size m is,

$$\begin{array}{ccccccc} Y_{[1]1} & Y_{[1]2} & \dots & Y_{[1]S_1} & & & \\ Y_{[2]1} & Y_{[2]2} & \dots & Y_{[2]S_2} & & & \\ \dots & \dots & \dots & \dots & \dots & & \\ Y_{[m]1} & Y_{[m]2} & \dots & Y_{[m]S_m} & & & \end{array}$$

If $S_1 = S_2 = \dots = S_m = n$, the RSS is said to be balanced, refers to an equal allocation of sample units to each of the rank order statistics, otherwise, it is said to be unbalanced.

In this dissertation, we propose to use a modified technique of the RR, considering a case when there is a variable of interest (Y) that is difficult to measure. However, an auxiliary covariate (X) that is highly correlated to the response variable is available and can be ranked easily. Assuming balanced RSS and ranking on (X), we have:

$$\begin{array}{l} (X_{(1)1}, Y_{[1]1}); (X_{(1)2}, Y_{[1]2}); \dots; (X_{(1)r}, Y_{[1]r}) \\ \\ (X_{(2)1}, Y_{[2]1}); (X_{(2)2}, Y_{[2]2}); \dots; (X_{(2)r}, Y_{[2]r}) \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (X_{(m)1}, Y_{[m]1}); (X_{(m)2}, Y_{[m]2}); \dots; (X_{(m)r}, Y_{[m]r}) \end{array},$$

where $[.]$ indicates imperfect ranking, while $(.)$ indicates perfect ranking. Let

$Y_{[1]1}, Y_{[2]1}, \dots, Y_{[m]1}, \dots, Y_{[m]r}$ be the j^{th} subsample and $\bar{Y}_{(j)}$ denote the subsample mean for the j^{th} judgment order statistics. Dell and Clutter (1972) proved that the sample mean based on RSS (\bar{Y}_{RSS}) is an unbiased estimator of the population mean μ_Y regardless of ranking errors, where

$$\bar{Y}_{RSS} = (mr)^{-1} \sum_{j=1}^m \sum_{k=1}^r Y_{[j]k}, \quad (3.3)$$

and $Y_{[j]k}$ is the j^{th} order statistics in the k^{th} cycle. Hence,

$$E(\bar{Y}_{RSS}) = (mr)^{-1} \sum_{j=1}^m \sum_{k=1}^r \mu_{Y_{[j]k}} = \mu_Y. \quad (3.4)$$

Based on the RSS properties, the variance of the \bar{X}_{RSS} is given by,

$$Var(\bar{Y}_{RSS}) = (mr)^{-2} \sum_{j=1}^m \sum_{k=1}^r \sigma_{Y_{[j]k}}^2 = \frac{\sigma_Y^2}{mr} - \frac{\sum_{j=1}^m \sum_{k=1}^r (\mu_{Y_{[j]k}} - \mu)^2}{(mr)^2}, \quad (3.5)$$

where,

$$\sigma_{Y_{[j]k}}^2 = V(Y_{[j]k}).$$

The asymptotically unbiased estimator of variance based on the RSS estimator is given by,

$$\hat{\sigma}_1^2 = (mr - 1)^{-1} \sum_{j=1}^m \sum_{k=1}^r (Y_{[j]k} - \bar{Y}_{RSS})^2. \quad (3.6)$$

Since both estimators (\bar{Y}_{RSS}) and (\bar{Y}_{SRS}) are unbiased we can compare their performances using the relative precision (RP) of an RSS estimator relative to the SRS estimator, where is defined as the ratio of the two variances for the corresponding estimators,

$$RP = \frac{VAR(\bar{Y}_{SRS})}{VAR(\bar{Y}_{RSS})}. \quad (3.7)$$

It can be shown that the bounds of this RP are $0 < RP < m$ where m is the set size. Indicating that, with appropriate unequal allocation, the RP may increase to a level of m . However, in the case of perfect ranking with equal allocations, it can be shown that the bounds of this RP are $1 \leq RP < \frac{m+1}{2}$. Since RP cannot be less than one, the RSS protocol cannot be worse than the SRS protocol.

Chapter 4

ON THE INFERENCE OF COVARIANCE ANALYSIS USING RANKED AUXILIARY COVARIATES

4.1. Introduction

The purpose of this dissertation is to investigate the performance of RSS when comparing L groups. The primary aim is to improve the statistical inference of group comparisons by incorporating one available continuous ranked auxiliary covariate, which is correlated with the variable of interest. The mean differences of the outcome between the groups are estimated by using the multiple regression approach in the ANCOVA models. Following the original procedure of RSS, we have proposed a modified RSS to randomly assign subjects into L groups. The proposed method is described in the next section.

4.2. The Modified Ranked Set Sampling Method

Assume a situation where the exact measurement of the outcome variable is difficult to obtain in terms of time and cost. Also, assume sampling units drawn from the population can be economically ranked by certain means without the actual measurement of the variable of interest. Suppose we have L experimental conditions, the procedure of selecting a RSS involves first selecting L SRS of subjects each of size m , where m is called a set size (note that m should be small, which is between 3 to 5 subjects). Therefore, the process of the modified RSS scheme to allocate subjects to the different groups can be performed as follows:

- 1) Randomly select (Lm) subjects identified from a population or a cohort, and partitioned into (L) independent sets each containing (m) subjects.
- 2) Within each selected set, rank the subjects with respect to one of the available auxiliary baseline covariates (X) and highly correlated with the response variable (Y) .
- 3) From each ranked sets, the subject with minimum rank of the value of (X) will be selected for study. Different experimental conditions (L) will be assigned randomly to the selected subjects.
- 4) The remaining $L(m-1)$ subjects from the selected set will be discarded. Repeat steps (1 - 3), however, in step 3, subjects with second minimum rank of the value of (X) will be selected for the study. The experimental conditions will be randomly assigned to the (L) selected subjects.
- 5) The process will continue in the same way until the subjects with rank m of (X) values are selected and randomly assigned to one of the (L) experimental conditions.

The above process represents a single RSS cycle for each experimental condition. The total sample size of the selected subjects is Lm from one cycle. However, if more subjects are needed, the whole process can be repeated (r) times. In this case, the total sample size of subjects selected for the entire process is $nL = Lmr$.

In term of notations, let $X_{i(j)k}$, i.e. $X_{i(1)k}, X_{i(2)k}, \dots, X_{i(m)k}$, where $i = 1, \dots, L$ experimental conditions, $j = 1, \dots, m$ subjects and $k = 1, \dots, r$ cycles, denote a random sample of subjects of size m based on the auxiliary variable (X) . Then, $X_{i(j)k}$ is the j^{th} ranked subject, from the i^{th} experimental condition in the k^{th} cycle.

Therefore,

$$\left\{ \begin{array}{l} \left(X_{i(j)k}, Y_{i[j]k} \right), \\ i = 1, \dots, L \\ j = 1, \dots, m \\ k = 1, \dots, r \end{array} \right\},$$

will denote the quantified data to be analyzed using ANCOVA model.

4.3. ANCOVA Models to Compare Groups

To simplify the notation, let (Y) be the response variable of interest, (X) is the auxiliary covariate, and (Z) is the indicator variable for experimental conditions. Let (Y) be a random variable with *c.d.f* $F_Y(y)$ and *p.d.f* $f_Y(y)$ with a finite mean μ_Y , and a finite variance σ_Y^2 . Then, $\{(Y_{i[1]1}, X_{i(1)1}, Z_i), \dots, (Y_{i[m]1}, X_{i(m)1}, Z_i); \dots; (Y_{i[1]r}, X_{i(1)r}, Z_i), \dots, (Y_{i[m]r}, X_{i(m)r}, Z_i)\}$ denote the corresponding of the selected RSS of size m and cycle size r assigned to i^{th} condition group. Therefore, the observed values of (Y, X, Z) of i^{th} condition can be written as,

$$y_{i[j]k} = \mu_0 + \beta_1 x_{i(j)k} + \gamma_i Z_i + \varepsilon_{i[j]k}, \quad (4.1)$$

$$i = 1, 2, \dots, L \quad \text{Experimental conditions}$$

$$j = 1, 2, \dots, m$$

$$k = 1, 2, \dots, r$$

where $[j]$ denotes the imperfect j^{th} ranking (ranking with errors). μ_0 is the overall mean, β_1 is the slop of (X) , γ_i is the i^{th} experiment group, and $\varepsilon_{i(j)k}$ is the random error.

In matrix notation, (4.1) can be written by,

$$Y_R = \beta X_R + \gamma Z_R + \varepsilon_R \quad (4.2)$$

where,

$$Y_R = \begin{bmatrix} y_{l[1]l} \\ y_{l[2]l} \\ \vdots \\ y_{l[m]l} \\ \vdots \\ y_{l[1]r} \\ y_{l[2]r} \\ \vdots \\ y_{l[m]r} \\ \vdots \\ y_{L[1]l} \\ y_{L[2]l} \\ \vdots \\ y_{L[m]l} \\ \vdots \\ y_{L[1]r} \\ y_{L[2]r} \\ \vdots \\ y_{L[m]r} \end{bmatrix}_{nL \times 1}, \quad X_R = \begin{bmatrix} 1 & x_{1(1)l} \\ 1 & x_{1(2)l} \\ \vdots & \vdots \\ 1 & x_{1(m)l} \\ \vdots & \vdots \\ 1 & x_{1(1)r} \\ 1 & x_{1(2)r} \\ \vdots & \vdots \\ 1 & x_{1(m)r} \\ \vdots & \vdots \\ 1 & x_{L(1)l} \\ 1 & x_{L(2)l} \\ \vdots & \vdots \\ 1 & x_{L(m)l} \\ \vdots & \vdots \\ 1 & x_{L(1)r} \\ 1 & x_{L(2)r} \\ \vdots & \vdots \\ 1 & x_{L(m)r} \end{bmatrix}_{nL \times 2},$$

Given that, $nL = Lmr$.

$\gamma = [\gamma_1, \gamma_2, \dots, \gamma_{L-1}]'$ is the vector of $L-1$ experiment group's effect, and ε is a vector of the random error term. Note that $\gamma_L = [-\gamma_1 - \gamma_2 \dots - \gamma_{L-1}]$. Also, let

$$Z = [I_1, I_2, \dots, I_{L-1}] \quad (4.3)$$

where,

$$\begin{aligned}
I_1 &= \begin{cases} 1 & \text{if case from treatment 1} \\ -1 & \text{if case from treatment } L \\ 0 & \text{Otherwise} \end{cases} \\
&\vdots \\
I_{L-1} &= \begin{cases} 1 & \text{if case from treatment } L-1 \\ -1 & \text{if case from treatment } L \\ 0 & \text{Otherwise} \end{cases}
\end{aligned} \tag{4.4}$$

are experimental condition indicators. In matrix notation, Z will become,

$$Z_R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ \hline \cdots & \cdots & \ddots & \cdots \\ \hline 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}_{nL \times (L-1)}$$

Therefore, the model (4.2) can be expressed as follows,

$$y_{i[j]k} = \beta_0 + \beta_1 x_{i(j)k} + \gamma_1 I_{ijk1} + \dots + \gamma_{L-1} I_{ijkL-1} + \varepsilon_{i[j]k} \tag{4.5}$$

where $\varepsilon_{i[j]k}$ is independent and normally distributed as $\varepsilon_{i[j]k} \sim N(0, \sigma_{[j]}^2)$, see (Stokes, 1977).

Since model (4.1) contains an intercept then, the design matrix (X) has a column of ones and can be written as $[I_n \quad \tilde{x}]$, and \tilde{x} denotes the centered observation $(X_{i(j)k} - \bar{X}_{i..})$, where

$$\bar{X}_{i..} = \sum_{j=1}^m \sum_{k=1}^r \frac{\bar{X}_{i(j)k}}{mr}, \quad i = 1, 2, \dots, L.$$

In the next section, we will develop theoretical aspects needed to analyze the ranked based data in ANCOVA models.

4.4. Some Asymptotic Results of Using the Modified Ranked Set Sampling

In the case of a SRS, when the values of the auxiliary covariates (X) in model (4.2) are assumed to be known and without errors, then the random errors, ε_{ijk} , are non-correlated random variables with mean 0 and variance σ^2 . If no replications are made, then the LSE have minimum variance and unbiased linear estimates of β and γ , by the Gauss-Markov theorem (Graybill, 1961; Rao, 1965). Furthermore, if the random errors ε_{ijk} are normally distributed, then the LSEs are also unbiased estimators with minimum variance.

However, if subjects j are selected into group i based on a RSS judgment, which then leads to the j^{th} order statistics, the random variable $Y_{i[j]k}$ has variance $\sigma_{Y_{[j]}}^2$. The differences between the expected mean of order statistics and population mean play an important role in RR because $\sigma_{Y_{[j]}}^2 = \sigma_Y^2 - \Delta_{Y_{[j]}}^2$ where $\Delta_{Y_{[j]}}^2 = (\mu_{Y_{[j]}} - \mu_Y)^2$. Let Y_1, \dots, Y_n be a simple random sample from a population with finite mean μ_Y and finite variance σ_Y^2 . Then the sample mean $\bar{Y} = \sum_{i=1}^n \frac{Y_i}{n}$ has an

expected value of $\mu_Y = E[\bar{Y}]$, and variance $V(\bar{Y}) = \frac{\sigma_Y^2}{n}$. Let $Y_{i[j]1}, \dots, Y_{i[j]r}$ be an independent

random sample of size (r) of the j^{th} order statistics in experimental group i , then the sample

mean is $\bar{Y}_{[j]} = \frac{1}{r} \sum_{k=1}^r Y_{[j]k}$. Therefore, $E[\bar{Y}_{[j]}] = \mu_{Y_{[j]}}$ and $V(\bar{Y}_{[j]}) = \frac{\sigma_{Y_{[j]}}^2}{r}$.

Using the above results, we need to prove the following proposition to show that the expected value of the k^{th} moment for the ordered population's outcome exists. In addition, using this proposition, we will show the asymptotic properties of the model's parameters estimators in ANCOVA model.

Proposition 1 (K^{th} moment): Let Y_1, \dots, Y_m be a random sample from a population with absolutely continuous *cdf* $F_Y(y)$, *pdf* $f_Y(y)$ and $E(|Y|^k) < \infty$, where $k \geq 1$. Then

$$E(|Y_{[j]}|^k) < \infty$$

Proof

By definition:

$$\begin{aligned} E(|Y_{[j]}|^k) &= \int_{-\infty}^{\infty} |Y|^k \frac{m!}{(j-1)!(m-j)!} [F(Y)]^{j-1} [1-F(Y)]^{m-j} f(Y) dy \\ &= \lim_{a \rightarrow \infty} \int_{-a}^a |Y|^k \frac{m!}{(j-1)!(m-j)!} [F(Y)]^{j-1} [1-F(Y)]^{m-j} f(Y) dy \end{aligned}$$

Since the sample size (m) is known and finite, $0 < [F(Y)] < 1$ and $0 < [1-F(Y)] < 1$, then

$$E(|Y_{[j]}|^k) = \frac{m!}{(j-1)!(m-j)!} \lim_{a \rightarrow \infty} \int_{-a}^a |Y|^k [F(Y)]^{j-1} [1-F(Y)]^{m-j} f(Y) dy <$$

$$\frac{m!}{(j-1)!(m-j)!} \lim_{a \rightarrow \infty} \int_{-a}^a |Y|^k f(Y) dy = \frac{m!}{(j-1)!(m-j)!} E(|Y|^k) < \infty$$

Therefore, $E(|Y_{[j]}|^k) < \infty$.

Hence, $E(|Y_{[j]}|^2) < \infty$,

and then, $V(Y_{[j]}) < \infty$.

Next, we show that $\sigma_{Y_{[j]}}^2$ has a consistent estimator.

Theorem 1: Let $Y_{[j]1}, Y_{[j]2}, \dots, Y_{[j]r}$ be a random sample from a population with *cdf* $F_{Y_{[j]}}(y)$,

pdf $f_{Y_{[j]}}(y)$, variance $V(Y_{[j]}) = \sigma_{Y_{[j]}}^2$ and finite fourth moment $(\mu_{4[j]})$ where $Y_{[j]}$ is the j^{th}

order statistics of a SRS of size m . Then the sample variance $S_{Y_{[j]}}^2$ is a consistent estimator of

the population variance $\sigma_{Y_{[j]}}^2$.

Proof

Given that, $\bar{Y}_{[j]} = \sum_{k=1}^r \frac{Y_{[j]k}}{r}$.

The sample variance is defined as,

$$\begin{aligned} S_{Y_{[j]}}^2 &= \frac{1}{r-1} \sum_{k=1}^r (Y_{[j]k} - \bar{Y}_{[j]})^2 \\ &= \frac{1}{r-1} \sum_{k=1}^r (Y_{[j]k} - \mu_{[j]} + \mu_{[j]} - \bar{Y}_{[j]})^2 \\ &= \frac{1}{r-1} \sum_{k=1}^r [(Y_{[j]k} - \mu_{[j]}) - (\bar{Y}_{[j]} - \mu_{[j]})]^2 \\ &= \frac{1}{r-1} \left[\sum_{k=1}^r (Y_{[j]k} - \mu_{[j]})^2 - 2 \sum_{k=1}^r (Y_{[j]k} - \mu_{[j]}) (\bar{Y}_{[j]} - \mu_{[j]}) + r (\bar{Y}_{[j]} - \mu_{[j]})^2 \right] \end{aligned}$$

$$= \frac{1}{r-1} \left[\sum_{k=1}^r (Y_{[j]k} - \mu_{[j]})^2 - r(\bar{Y}_{[j]} - \mu_{[j]})^2 \right]$$

First, we need to show that $\bar{Y}_{[j]} \xrightarrow{P} \mu_{Y_{[j]}}$.

For any $\varepsilon > 0$ and using Chebychav's inequality,

$$P\left(|\bar{Y}_{[j]} - \mu_{Y_{[j]}}| > \varepsilon\right) \leq \frac{\sigma_{Y_{[j]}}^2}{r\varepsilon^2} \rightarrow 0 \text{ as } r \rightarrow \infty, \text{ by proposition 1.}$$

Hence, $\bar{Y}_{[j]} \xrightarrow{P} \mu_{Y_{[j]}}$

On the other hand, given $E(|Y_{[j]}|^4) < \infty$ and $E(Y_{[j]}) = \mu_{[j]}$ as seen in proposition 1, it can be shown by using the Strong Law of Large Numbers that $\bar{Y}_{[j]} \xrightarrow{a.s.} \mu_{Y_{[j]}}$ (Rohatgi & Saleh, 2001).

Now,

$$\begin{aligned} E(S_{Y_{[j]}}^2) &= \frac{1}{r-1} \left[\sum_{k=1}^r E(Y_{[j]k} - \mu_{[j]})^2 - rE(\bar{Y}_{[j]} - \mu_{[j]})^2 \right] \\ &= \frac{1}{r-1} [r\sigma_{[j]}^2 - r\frac{\sigma_{[j]}^2}{r}] = \sigma_{Y_{[j]}}^2 \end{aligned}$$

Hence, $S_{Y_{[j]}}^2$ is an unbiased estimator of $\sigma_{Y_{[j]}}^2$

To show that $S_{[j]}^2 \xrightarrow{P} \sigma_{[j]}^2$, it is sufficient to show that $V(S_{[j]}^2) \rightarrow 0$ as $r \rightarrow \infty$.

By using corollary 7.3.2 from (Rohatgi & Saleh, 2001),

$$V(S_{Y_{[j]}}^2) = \frac{\mu_{4[j]}}{r} + \frac{(3-r)\mu_{2[j]}^2}{r(r-1)} \text{ where the fourth order statistics moment } \mu_{4[j]} \text{ exists by using}$$

proposition 1, then $V(S_{Y_{[j]}}^2) \rightarrow 0$. Therefore, $S_{Y_{[j]}}^2 \xrightarrow{P} \sigma_{Y_{[j]}}^2$.

Using the above results, next we derive ANCOVA model parameters estimations by using ranked auxiliary covariate to compare between L groups.

4.5. Estimating Unknown Parameters: Weighted Least Squares

We can rewrite the model (4.2) as,

$$Y_R = M_R \theta + \varepsilon_R, \quad (4.6)$$

where, $M_R = [X_R \quad Z_R]'$, $\theta = [\beta \quad \gamma]'$ and ε_R is a random error term.

The method of OLS is not recommended for estimation in this case because the variance of $Y_{[j]}$ is not a constant. Therefore, a weighted least squared estimation method is recommended as an alternative to OLS for stabilizing the variances of the error terms. The WLS estimator is a useful method for estimating the model parameters when the observed values of variability are different over the predictor values of the ranked subjects, where the weights are determined using the variance of the order statistics. The WLS is different from OLS in that it includes an additional weight for each term in the model to determine how much each ranked subject in the data set influences the parameter estimates. Given that, $V(Y_{[j]}) = \sigma_{Y_{[j]}}^2$, let

$$\Sigma = \begin{bmatrix} \sigma^2_{l[1]l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_{l[2]l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sigma^2_{l[m]l} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{2[1]l} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{L[m]l} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{l[1]r} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{l[2]r} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \sigma^2_{L[m]r} \end{bmatrix}_{nL \times nL}$$

be the variance-covariance matrix of Y_R . The most common and straightforward weighting scheme can be defined as $w_j = 1/\sigma_{Y_{[j]}}$. However, $\sigma_{Y_{[j]}}^2$ is unknown, so the estimated weight is given by, $\hat{w}_j = 1/S_{Y_{[j]}}$. Therefore, the diagonal of the estimated weighting coefficient matrix is given as follows, let $\hat{W} = \hat{\Sigma}^{-1/2}$, where

$$\hat{\Sigma}^{-1/2} = \begin{bmatrix} 1/S_{l[1]l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1/S_{l[2]l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 1/S_{l[m]l} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1/S_{2[l]l} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/S_{L[m]l} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/S_{l[1]r} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/S_{l[2]r} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1/S_{L[m]r} \end{bmatrix}_{nL \times nL}$$

Model (4.6) can be rewritten as:

$$\hat{W}Y_R = \hat{W}M_R\theta + \hat{W}\varepsilon_R. \quad (4.7)$$

The model (4.7) can be written as,

$$V = R\theta + \varepsilon^*, \quad (4.8)$$

where $V = \hat{W}Y_R$, $R = \hat{W}M_R$, and $\varepsilon^* = \hat{W}\varepsilon_R$.

The vector of auxiliary covariates and group indicator (M) is assumed to be known and available to a finite number of parameters $\theta = (\theta_1, \dots, \theta_{p-1})'$, where $p < n$. We estimate the

parameter θ by the value $\hat{\theta}$ that gives the best fit to the model. The weighted least square estimator, denoted by $\hat{\theta}$, is that value of θ that minimizes the sum square of errors. Denote the squared length of an n-dimensional vector ε by $\|\varepsilon^*\|^2 = \varepsilon^{*'} \varepsilon$. Then it is easy to see that in weighted least squares, we are minimizing,

$$\varepsilon^{*'} \varepsilon^* = (V - R\theta)' (V - R\theta) , \quad (4.9)$$

given that,

$$\hat{\theta} = (R'R)^{-1} R'V$$

To find the values of SSE, parameter estimator of $\hat{\theta}$ includes the ranked auxiliary covariates effects $\hat{\beta}$ and treatment effects $\hat{\gamma}$, we will use the sweep operator method as follows:

Let $C = \hat{W}'\hat{W}$, the first sweep operation step $A^{(0)}$, is given by,

$$\begin{aligned}
A^{(0)} &= \left[\begin{array}{c|c} R'R & R'V \\ \hline V'R & V'V \end{array} \right] \\
&= \left[\begin{array}{c|c} M'CM & M'CY_R \\ \hline Y'_R CM & Y'_R CY_R \end{array} \right] \\
&= \left[\begin{array}{cc|c} X'_R CX_R & X'_R CZ_R & X'_R CY_R \\ \hline Z'_R CX_R & Z'_R CZ_R & Z'_R CY_R \\ \hline Y'_R CX_R & Y'_R CZ_R & Y'_R CY_R \end{array} \right]
\end{aligned}$$

Sweeping $A^{(0)}$ on the columns associated with $X'_R CX_R$ yields,

$$A^{(1)} = SWP(A,1) = \left[\begin{array}{cc|c} (X'_R CX_R)^{-1} & (X'_R CX_R)^{-1} X'_R CZ_R & (X'_R CX_R)^{-1} X'_R CY_R \\ \hline Z'_R CX_R (X'_R CX_R)^{-1} & Z'_R CZ_R - Z'_R CX_R (X'_R CX_R)^{-1} X'_R CZ_R & Z'_R CY_R - Z'_R CX_R (X'_R CX_R)^{-1} X'_R CY_R \\ \hline -Y'_R CX_R (X'_R CX_R)^{-1} & Y'_R CZ_R - Y'_R CX_R (X'_R CX_R)^{-1} X'_R CZ_R & Y'_R CY_R - Y'_R CX_R (X'_R CX_R)^{-1} X'_R CY_R \end{array} \right]$$

Let

$$H = \hat{W}X_R(X_R'CX_R)^{-1}X_R'\hat{W}',$$

then,

$$Z_R' CZ_R - Z_R' CX_R (X_R' CX_R)^{-1} X_R' CZ_R = Z_R' \hat{W}' (I - H) \hat{W} Z_R$$

$$Z_R' CY_R - Z_R' CX_R (X_R' CX_R)^{-1} X_R' CY_R = Z_R' \hat{W}' (I - H) \hat{W} Y_R$$

$$Y_R' CZ_R - Y_R' CX_R (X_R' CX_R)^{-1} X_R' CZ_R = Y_R' \hat{W}' (I - H) \hat{W} Z_R$$

$$Y_R' CY_R - Y_R' CX_R (X_R' CX_R)^{-1} X_R' CY_R = Y_R' \hat{W}' (I - H) \hat{W} Y_R$$

Hence, the same sweep operator $A^{(1)}$ can be rewritten as,

$$A^{(1)} = SWP(A, 1) = \left[\begin{array}{cc|c} (X_R' CX_R)^{-1} & (X_R' CX_R)^{-1} X_R' CZ_R & (X_R' CX_R)^{-1} X_R' CY_R \\ \hline Z_R' CX_R (X_R' CX_R)^{-1} & Z_R' \hat{W}' (I - H) \hat{W} Z_R & Z_R' \hat{W}' (I - H) \hat{W} Y_R \\ \hline -Y_R' CX_R (X_R' CX_R)^{-1} & Y_R' \hat{W}' (I - H) \hat{W} Z_R & Y_R' \hat{W}' (I - H) \hat{W} Y_R \end{array} \right]$$

Note that the $SSE_{(\text{Reduced})}$ of the reduced model is given by,

$$SSE_{(\text{Reduced})}(X_R) = Y_R' \hat{W}' (I - H) \hat{W} Y_R, \quad (4.10)$$

and

$$\hat{\beta}_{(\text{Reduced})} = (X_R' CX_R)^{-1} X_R' CY_R. \quad (4.11)$$

Sweeping $A^{(1)}$ on the columns associated with $Z_R' \hat{W}' (I - H) \hat{W} Z_R$ yields,

$$A^{(2)} = SWP(A, 2) =$$

$$\left[\begin{array}{cc|c} (X'_R CX_R)^{-1} [I + X'_R CZ_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R CX_R (X'_R CX_R)^{-1}] & -(X'_R CX_R)^{-1} X'_R CZ_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} & \hat{\beta}_{(\text{Reduced})} - (X'_R CX_R)^{-1} X'_R CZ_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R \hat{W}'(I-H) \hat{W} Y_R \\ \\ -(Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R CX_R (X'_R CX_R)^{-1} & (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} & (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R \hat{W}'(I-H) \hat{W} Y_R \\ \hline Y'_R CX_R (X'_R CX_R)^{-1} - Y'_R \hat{W}'(I-H) \hat{W} Z_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R CX_R (X'_R CX_R)^{-1} & -Y'_R \hat{W}'(I-H) \hat{W} Z_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} & Y'_R \hat{W}'(I-H) \hat{W} Y_R - Y'_R \hat{W}'(I-H) \hat{W} Z_R (Z'_R \hat{W}'(I-H) \hat{W} Z_R)^{-1} Z'_R \hat{W}'(I-H) \hat{W} Y_R \end{array} \right]$$

Note that the $SSE_{(Full)}$ of the full model is given by,

$$SSE_{(Full)}(X_R, Z_R) = Y_R' \hat{W}'(I - H) \hat{W} Y_R - Y_R' \hat{W}'(I - H) \hat{W} Z_R (Z_R' \hat{W}'(I - H) \hat{W} Z_R)^{-1} Z_R' \hat{W}'(I - H) \hat{W} Y_R, \quad (4.12)$$

and the estimated unknown parameters are given by,

$$\gamma_{RSS} = (Z_R' \hat{W}'(I - H) \hat{W} Z_R)^{-1} Z_R' \hat{W}'(I - H) \hat{W} Y_R, \quad (4.13)$$

and

$$\begin{aligned} \hat{\beta}_{RSS} &= (X_R' C X_R)^{-1} X_R' C Y_R - (X_R' C X_R)^{-1} X_R' C Z_R (Z_R' \hat{W}'(I - H) \hat{W} Z_R)^{-1} Z_R' \hat{W}'(I - H) \hat{W} Y_R \\ &= \hat{\beta}_{(Reduced)} - (X_R' C X_R)^{-1} X_R' C Z_R \gamma_{RSS} \end{aligned} \quad (4.14)$$

Next, we need to show that the estimated unknown parameters $\hat{\beta}_{RSS}$ and $\hat{\gamma}_{RSS}$ are unbiased estimators for β and γ , by using the following theorem.

Theorem 2:

Let $\hat{W} Y_R = \hat{W} X_R \beta + \hat{W} Z_R \gamma + \hat{W} \varepsilon_R$ where X_R is an $nL \times p$, Z_R is an $nL \times (L-1)$ matrices of full rank; β and γ are $[p \times 1]$ and $[(L-1) \times 1]$ respectively vectors of unknown parameters; ε is an $n \times 1$ random errors vector with mean and variance defined by order statistics $(0, \sigma_{Y_{(j)}}^2)$. The weighted least squares estimators $\hat{\beta}_{RSS}$ and $\hat{\gamma}_{RSS}$ are asymptotically unbiased estimators for β and γ respectively.

Proof

Given that,

$$\hat{\beta}_{RSS} = (X_R' C X_R)^{-1} X_R' C Y_R - (X_R' C X_R)^{-1} X_R' C Z_R (Z_R' \hat{W}'(I - H) \hat{W} Z_R)^{-1} Z_R' \hat{W}'(I - H) \hat{W} Y_R,$$

for large sample size, the estimated weighted variance-covariance matrix can be written as

$\hat{W}' \cong W' = \Sigma^{-\frac{1}{2}}$, then we show that the expected value of $\hat{\beta}_{RSS}$ is given by,

$$\begin{aligned}
E(\hat{\beta}_{RSS}) &= \left[(X'_R C X_R)^{-1} X'_R C - (X'_R C X_R)^{-1} X'_R C Z_R (Z'_R C W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] E(Y_R) \\
&= \left[(X'_R C X_R)^{-1} X'_R C - (X'_R C X_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] [X_R \beta + Z_R \gamma] \\
&= (X'_R C X_R)^{-1} X'_R C X_R \beta - (X'_R C X_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W X_R \beta + \\
&\quad (X'_R C X_R)^{-1} X'_R C Z_R \gamma - (X'_R C X_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W Z_R \gamma \\
&= \beta - 0 + (X'_R C X_R)^{-1} X'_R C Z_R \gamma - (X'_R C X_R)^{-1} X'_R C Z_R \gamma \\
&= \beta.
\end{aligned}$$

Moreover, given that,

$$\hat{\gamma}_{RSS} = (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W Y_R,$$

then the expected value of $\hat{\gamma}_{RSS}$ is given by,

$$\begin{aligned}
E(\hat{\gamma}_{RSS}) &= \left[(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] E(Y_R) \\
&= \left[(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] [(X_R \beta + Z_R \gamma)] \\
&= (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W X_R \beta + (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W Z_R \gamma \\
&= \gamma.
\end{aligned}$$

Now,

since $V(W Y_R) = W' \Sigma W = I$, the variances of the unknown parameters are given by,

$$\begin{aligned}
V(\hat{\beta}_{RSS}) &= \left[(X'_R CX_R)^{-1} X'_R C - (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] V(Y_R) \\
&\quad \left[(X'_R CX_R)^{-1} X'_R C - (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right]' \\
&= \left[(X'_R CX_R)^{-1} X'_R C - (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right] [\Sigma] \\
&\quad \left[(X'_R CX_R)^{-1} X'_R C - (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W \right]' \\
&= [(X'_R CX_R)^{-1} X'_R C X_R (X'_R CX_R)^{-1} - (X'_R CX_R)^{-1} X'_R C W' (I - H) W Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R C X_R (X'_R CX_R)^{-1} - \\
&\quad (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W X_R (X'_R CX_R)^{-1} + \\
&\quad (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R C X_R (X'_R CX_R)^{-1} \\
&= (X'_R CX_R)^{-1} + (X'_R CX_R)^{-1} X'_R C Z_R (Z'_R W' (I - H) W Z_R)^{-1} Z'_R C X_R (X'_R CX_R)^{-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
V(\hat{\gamma}_{RSS}) &= [(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W] V(Y_R) [(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W]' \\
&= [(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W] [\Sigma] [(Z'_R W' (I - H) W Z_R)^{-1} Z'_R W' (I - H) W]' \\
&= [(Z'_R W' (I - H) W Z_R)^{-1}].
\end{aligned}$$

For testing the null hypothesis $H_0 : \gamma_1 = \gamma_2 = \dots = \gamma_L = 0$, we need to show the following theorem.

Theorem 3: Let the reduced model $\hat{W}Y = \hat{W}X_R\beta + \hat{W}\varepsilon$ where X_R is $n \times p$ of rank, $E(\varepsilon) = 0$

and $Var(\varepsilon) = \Sigma$. Let $s_1^2 = \frac{SSE_{(Reduced)}(\mathbf{X}_R)}{nL - p}$, then s_1^2 is an unbiased estimator for $W'\Sigma W$

Proof

By definition,

$$E(s_1^2) = E\left[\frac{SSE_{(Reduced)}(\mathbf{X}_R)}{nL - p}\right] = \frac{1}{nL - p} E[SSE_{(Reduced)}(\mathbf{X}_R)],$$

the sum squares of errors from the reduced model is given by,

$$SSE_{(Reduced)}(\mathbf{X}_R) = Y_R' \hat{W}' (I - H) \hat{W} Y_R$$

let, $A = W'(I - H)W$, where $H = WX(X'X)^{-1}X'W'$ is symmetric and idempotent, then

$$\begin{aligned} E[SSE_{(Reduced)}(\mathbf{X}_R)] &= E[Y_R' A Y_R] \\ &= tr(A\Sigma) + E[Y_R'] A E[Y_R] \\ &= tr[W(I - H)W'\Sigma] + [\beta' X_R' [W(I - H)W'] X_R \beta], \\ &= tr[(I - H)W'\Sigma W] + 0 \\ &= (nL - p), \end{aligned}$$

for large n , $\hat{W}' \cong W' = \Sigma^{-\frac{1}{2}}$,

and hence, S_1^2 is an unbiased estimator for the variance $W'\Sigma W$.

Under the normality assumption of Y , it can be shown that $SSE_{(Reduced)}(X_R)$ follows a chi-square distribution with $(nL - p)$ degrees of freedom, i.e. $SSE_{(Reduced)}(\mathbf{X}_R) \sim \chi_{(nL-p)}^2$.

Similarly, in the full model, s_2^2 is given by,

$$s_2^2 = \frac{SSE_{(full)}(X_R, Z_R)}{(nL - p - L + 1)}.$$

The next theorem shows that s_2^2 for the full model is an unbiased estimator for $W'\Sigma W$.

Theorem 4: Let $\hat{W}Y_R = \hat{W}X_R\beta + \hat{W}Z_R\gamma + \hat{W}\varepsilon$ where X_R is an $nL \times p$ and Z_R is an $nL \times (L-1)$ matrices of full rank, β is a $p \times 1$ vector and γ is an $(L-1) \times 1$ vector of unknown parameters, and ε^* is an $nL \times 1$ normally distributed random vector with $E(\varepsilon^*) = 0$. Let

$$s_2^2 = \frac{SSE_{(full)}(X_R, Z_R)}{(nL - p - L + 1)}. \text{ Then } S_2^2 \text{ is an unbiased estimator for } W'\Sigma W$$

Proof

By definition,

$$E(S_2^2) = E\left[\frac{SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R)}{nL - p - L + 1}\right].$$

Now, let $A = \hat{W}'(I - H)\hat{W}$, where $H = \hat{W}X(X'X)^{-1}X'\hat{W}'$ is symmetric and idempotent matrix,

then the sum squares of errors from the full model form is given by,

$$SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R) = Y_R'AY_R - Y_R'AZ_R(Z_R'AZ_R)^{-1}Z_R'AY_R.$$

Thus,

$$E[SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R)] = E[Y_R'AY_R] - E[Y_R'AZ_R(Z_R'AZ_R)^{-1}Z_R'AY_R],$$

where,

$$\begin{aligned} E[Y_R'AY_R] &= tr(A\Sigma) + E[Y_R']AE[Y_R] \\ &= tr[(I - H)\hat{W}'\Sigma\hat{W}] + [(\beta'X_R' + \gamma'Z_R')\hat{W}(I - H)\hat{W}'][X_R\beta + Z_R\gamma], \\ &= (nL - p) + \gamma'Z_R'AZ_R\gamma \end{aligned}$$

for large n, $\hat{W}' \cong W' = \Sigma^{-\frac{1}{2}}$, and,

$$\begin{aligned}
E\left[Y_R'AZ_R(Z_R'AZ_R)^{-1}Z_R'AY_R\right] &= \text{tr}\left[(I-H)\hat{W}Z_R(Z_R'AZ_R)^{-1}Z_R'\hat{W}'(I-H)\hat{W}\Sigma\hat{W}'\right] + \\
&\quad \left[\beta'X_R' + \gamma'Z_R'\right]\left[\hat{W}(I-H)\hat{W}'\right]\left[X_R\beta + Z_R\gamma\right] \\
&= (L-1) + \gamma'Z_R'AZ_R\gamma,
\end{aligned}$$

then, the expected value of the full rank SSE is given by,

$$\begin{aligned}
E\left[SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R)\right] &= [(n-p) + \gamma'Z_R'AZ_R\gamma] - [(L-1) + \gamma'Z_R'AZ_R\gamma] \\
&= (nL-p) - (L-1) = nL-p-L+1.
\end{aligned}$$

Therefore, S_2^2 is an unbiased estimator for the variance $W'\Sigma W$.

Again under the normality assumption of Y , $SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R)$ approximately follows a chi-

square distribution with $nL-p-L+1$ degrees of freedom, i.e. $SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R) \sim \chi_{(nL-p-L+1)}^2$.

Using the above results, the differences between the reduced model error sum of squares,

$SSE_{(reduced)}(\mathbf{X}_R)$, and the full model error sum of squares, $SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R)$, is given by

$$\begin{aligned}
SSR(\mathbf{Z}_R | \mathbf{X}_R) &= SSE_{(reduced)}(\mathbf{X}_R) - SSE_{(full)}(\mathbf{X}_R, \mathbf{Z}_R) \\
&= [Y_R'AY_R] - [Y_R'AY_R - Y_R'AZ_R(Z_R'AZ_R)^{-1}Z_R'AY_R] \\
&= Y_R'AZ_R(Z_R'AZ_R)^{-1}Z_R'AY_R,
\end{aligned}$$

and under the null hypothesis $H_0: \gamma_1 = \gamma_2 = \dots = \gamma_L = 0$

$$E[SSR(\mathbf{Z}_R | \mathbf{X}_R)] = [(nL-p)] - [(nL-p-L+1)] = (L-1).$$

Hence, under the null hypothesis $SSR(\mathbf{Z}_R | \mathbf{X}_R)$ approximately follows a chi-squar, i.e.

$$SSR(\mathbf{Z}_R | \mathbf{X}_R) \sim \chi_{(L-1)}^2.$$

4.6. Hypothesis Testing

For the fixed effect model (4.1), the hypothesis testing for treatment effects involved in this model can be written as follows,

$$H_0 : \gamma_1 = \gamma_2 = \dots = \gamma_L = 0$$

$$H_1 : \text{not all } \gamma_i \text{ equal zero}$$

The reduced model under H_0 is given by:

$$V = \hat{W}X_R\beta + \varepsilon^*, \quad (4.15)$$

the sum squares of errors for the reduced model is given by:

$$SSE(R) = SSE_{(Reduced)}(X_R), \quad (4.16)$$

However, the full model is given by

$$V = R\theta + \varepsilon^*, \quad (4.17)$$

the sum squares of errors of the full model is given by:

$$SSE(F) = SSE_{(full)}(X_R, Z_R) \quad (4.18)$$

Consequently, the test involves for the null hypothesis for testing

$$H_0 : \gamma_1 = \gamma_2 = \dots = \gamma_L = 0, \text{ is given by}$$

$$\begin{aligned} F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\ &= \frac{SSR(Z_R | X_R) / (L-1)}{SSE(F) / (nL - p - L + 1)}, \end{aligned} \quad (4.19)$$

has approximately F-distribution with $(L-1, nL - p - L + 1)$ degrees of freedom,

Chapter 5

A SIMULATION STUDY

5.1. Design of the Simulation Study

A simulation study is conducted to investigate the performance of the modified RSS method when applied to L groups for comparison. In this simulation study, the empirical nominal values and the empirical power values are estimated for the modified RSS and compared to the SRS. A multiple regression approach has been used in an ANCOVA model to compare the average mean differences of the continuous outcome of interest (Y) between the selected groups (Z). The auxiliary covariate (X) is assumed to be a continuous variable, and ranked based on an order statistics method. For each group, samples were selected by set size of ($m = 3, 4 \text{ and } 5$) units and cycles ($k = 10 \text{ and } 30$).

For accurate comparison, the samples of RSS and SRS, within each group were selected with the same sample sizes, ($n = 30, 40, 50, 90, 120 \text{ and } 150$) where $n = Lmr$. The level of significance $\alpha = 0.05$ is considered. Different correlation coefficients between the auxiliary covariate (X) and the variable of interest (Y) are considered to be ($\rho = \pm 0.3, \pm 0.5, \pm 0.8$). The differences in means between groups ($d = 0, 0.3, 0.5, 0.8$) are considered for two experimental groups; however, the differences in means of ($d_1 = 0, 0.3, 0.5, 0.8$) and ($d_2 = 0, 0.3, 1$) are considered for three experimental groups. The normal distribution is used in the simulation, and all simulation results are calculated based on 5,000 replicates.

5.2 Results of Simulation

Table 1 presents the estimated probability of the type I error under the null hypothesis of no mean differences between the two groups, for RSS and SRS. In general, both sampling techniques RSS and SRS give close estimates to the nominal value ($\alpha = 0.05$) for all suggested set sizes, sample sizes, and correlation coefficient values. However, when using RSS in ANCOVA models to compare the groups' effects, results show that the α for the modified RSS is smaller than the SRS for different sample sizes and different correlation coefficients.

Table 1: Empirical Nominal Value ($\alpha = 0.05$) For Two Interventional Groups.

| | | α | | | | | |
|-----|-------------|---------------------------------|--------|---------|--------|---------|--------|
| | | $H_0 : \gamma_1 = \gamma_2 = 0$ | | | | | |
| | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 10 | 0.3 | 0.041 | 0.0542 | 0.0474 | 0.0498 | 0.0462 | 0.0494 |
| 30 | | 0.034 | 0.049 | 0.0337 | 0.0483 | 0.0376 | 0.0536 |
| 10 | 0.5 | 0.0412 | 0.0476 | 0.0444 | 0.0524 | 0.0448 | 0.0448 |
| 30 | | 0.0410 | 0.0494 | 0.0410 | 0.0494 | 0.0366 | 0.0448 |
| 10 | 0.9 | 0.0388 | 0.0472 | 0.0496 | 0.0562 | 0.0478 | 0.0542 |
| 30 | | 0.0418 | 0.0452 | 0.0442 | 0.0550 | 0.0440 | 0.0484 |
| 10 | -0.3 | 0.0316 | 0.0362 | 0.0441 | 0.0478 | 0.0475 | 0.0528 |
| 30 | | 0.0378 | 0.0482 | 0.0402 | 0.0503 | 0.0414 | 0.0452 |
| 10 | -0.5 | 0.0404 | 0.0496 | 0.0390 | 0.0564 | 0.0440 | 0.0476 |
| 30 | | 0.0418 | 0.0470 | 0.0424 | 0.0522 | 0.0442 | 0.0470 |
| 10 | -0.9 | 0.0500 | 0.0508 | 0.0458 | 0.0524 | 0.0544 | 0.0558 |
| 30 | | 0.0390 | 0.0498 | 0.0384 | 0.0536 | 0.0398 | 0.0510 |

Tables 2-4 show the estimated empirical power under different alternative hypotheses, $(H_A : \gamma_1 \neq \gamma_2)$ for comparing two experimental groups. In general, these tables show that using RSS scheme is more powerful than using SRS method for all given correlation coefficients. The estimated power of the alternative hypothesis testing obtained by using RSS increases as the mean difference (d), sample size, and correlation coefficient increases more than the estimated power obtained by using SRS. However, the results are more powerful for positive correlation coefficients than the negative for both sampling schemes.

Table 2: Empirical Power For Testing Two Interventional Groups With Mean Different ($d=0.3$).

| | | | $1 - \beta$ | | | | | |
|-----|-----|-------------|--|--------|---------|--------|---------|--------|
| | | | H_A : at least one γ not equal zero | | | | | |
| | | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| d | r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.3 | 10 | 0.3 | 0.0692 | 0.0690 | 0.0892 | 0.0760 | 0.1048 | 0.0788 |
| | 30 | | 0.1080 | 0.1072 | 0.1530 | 0.1258 | 0.2124 | 0.1460 |
| | 10 | 0.5 | 0.0734 | 0.0654 | 0.0822 | 0.0716 | 0.1056 | 0.0872 |
| | 30 | | 0.1270 | 0.1176 | 0.1752 | 0.1388 | 0.2410 | 0.1660 |
| | 10 | 0.9 | 0.1646 | 0.1304 | 0.2358 | 0.1718 | 0.3202 | 0.1996 |
| | 30 | | 0.4208 | 0.3382 | 0.5892 | 0.4198 | 0.7412 | 0.4934 |
| | 10 | -0.3 | 0.0678 | 0.0632 | 0.0820 | 0.0708 | 0.0952 | 0.0704 |
| | 30 | | 0.0984 | 0.0974 | 0.1348 | 0.1166 | 0.1872 | 0.1372 |
| | 10 | -0.5 | 0.0710 | 0.0612 | 0.0822 | 0.0670 | 0.0978 | 0.0690 |
| | 30 | | 0.0972 | 0.0902 | 0.1262 | 0.0986 | 0.1584 | 0.1176 |
| | 10 | -0.9 | 0.0600 | 0.0632 | 0.0712 | 0.0644 | 0.0868 | 0.0636 |
| | 30 | | 0.0700 | 0.0796 | 0.1048 | 0.0868 | 0.1244 | 0.0976 |

Table 3: Empirical Power For Testing Two Interventional Groups With Mean Different ($d=0.5$).

| $1 - \beta$ | | | | | | | | |
|--|-----|-------------|---------|--------|---------|--------|---------|--------|
| H_A : at least one γ not equal zero | | | | | | | | |
| d | r | ρ | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| | | | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.5 | 10 | 0.3 | 0.1110 | 0.0962 | 0.1572 | 0.1254 | 0.2046 | 0.1412 |
| | 30 | | 0.2542 | 0.2090 | 0.3820 | 0.2752 | 0.5298 | 0.3278 |
| | 10 | 0.5 | 0.1302 | 0.1106 | 0.1624 | 0.1392 | 0.2310 | 0.161 |
| | 30 | | 0.3032 | 0.2494 | 0.4384 | 0.3202 | 0.5996 | 0.3800 |
| | 10 | 0.9 | 0.3686 | 0.2992 | 0.5452 | 0.3886 | 0.6986 | 0.4680 |
| | 30 | | 0.8328 | 0.7268 | 0.9416 | 0.8334 | 0.9940 | 0.9132 |
| | 10 | -0.3 | 0.1052 | 0.0906 | 0.1418 | 0.1030 | 0.1858 | 0.1232 |
| | 30 | | 0.2160 | 0.1808 | 0.3176 | 0.2388 | 0.4512 | 0.2746 |
| | 10 | -0.5 | 0.0994 | 0.0860 | 0.1198 | 0.1042 | 0.1672 | 0.1074 |
| | 30 | | 0.2038 | 0.1722 | 0.2862 | 0.2156 | 0.3982 | 0.254 |
| | 10 | -0.9 | 0.0898 | 0.0732 | 0.1136 | 0.0840 | 0.1356 | 0.0868 |
| | 30 | | 0.1576 | 0.1264 | 0.2104 | 0.1584 | 0.2846 | 0.1894 |

Table 4: Empirical Power For Testing Two Interventional Groups With Mean Different ($d=0.8$).

| | | | $1 - \beta$ | | | | | |
|-----|-----|-------------|--|--------|---------|--------|---------|--------|
| | | | H_A : at least one γ not equal zero | | | | | |
| | | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| d | r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.8 | 10 | 0.3 | 0.2284 | 0.1740 | 0.3376 | 0.2292 | 0.4594 | 0.2786 |
| | 30 | | 0.5696 | 0.4502 | 0.7706 | 0.5844 | 0.9030 | 0.6686 |
| | 10 | 0.5 | 0.2532 | 0.2168 | 0.4026 | 0.2698 | 0.5294 | 0.3240 |
| | 30 | | 0.6528 | 0.5156 | 0.8540 | 0.6666 | 0.9504 | 0.7588 |
| | 10 | 0.9 | 0.7560 | 0.6286 | 0.9012 | 0.7558 | 0.9754 | 0.8588 |
| | 30 | | 0.9974 | 0.9780 | 0.9998 | 0.9974 | 1 | 0.9998 |
| | 10 | -0.3 | 0.2102 | 0.1606 | 0.2896 | 0.1914 | 0.3982 | 0.2436 |
| | 30 | | 0.5040 | 0.4006 | 0.7026 | 0.5116 | 0.8592 | 0.5908 |
| | 10 | -0.5 | 0.1778 | 0.1536 | 0.2596 | 0.1884 | 0.3702 | 0.2160 |
| | 30 | | 0.4564 | 0.3614 | 0.6416 | 0.4528 | 0.8024 | 0.5472 |
| | 10 | -0.9 | 0.1434 | 0.1158 | 0.2038 | 0.1440 | 0.2654 | 0.1618 |
| | 30 | | 0.3166 | 0.2624 | 0.4810 | 0.3320 | 0.6356 | 0.3968 |

Similarly, Table 5 shows the empirical nominal value for the mean differences between three experimental groups, using RSS and SRS. In general, both sampling schemes of RSS and SRS give close estimates to the nominal value ($\alpha = 0.05$) for all suggested set sizes, sample sizes, and correlation coefficient values. However, results show that the nominal value (α) for the modified RSS is smaller than the SRS for different sample sizes and different correlation coefficients.

Table 5: Empirical Nominal Value ($\alpha = 0.05$) For Three Interventional Groups.

| α | | | | | | | |
|--|-------------|---------------|---------------|---------------|---------------|---------|--------|
| $H_0 : \gamma_1 = \gamma_2 = \gamma_3 = 0$ | | | | | | | |
| r | ρ | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| | | RSS | SRS | RSS | SRS | RSS | SRS |
| 10 | 0.3 | 0.0434 | 0.0516 | 0.0476 | 0.0506 | 0.0414 | 0.0452 |
| 30 | | 0.0392 | 0.0508 | 0.0346 | 0.0498 | 0.0374 | 0.0478 |
| 10 | 0.5 | 0.0388 | 0.0520 | 0.0430 | 0.0530 | 0.0414 | 0.0484 |
| 30 | | 0.0346 | 0.0520 | 0.0334 | 0.0542 | 0.0360 | 0.0466 |
| 10 | 0.9 | 0.0444 | 0.0490 | 0.0480 | 0.0502 | 0.0464 | 0.0484 |
| 30 | | 0.0338 | 0.0488 | 0.0422 | 0.0526 | 0.0436 | 0.0474 |
| 10 | -0.3 | 0.0542 | 0.0538 | 0.0544 | 0.0520 | 0.0422 | 0.0472 |
| 30 | | 0.0378 | 0.0486 | 0.0396 | 0.0492 | 0.0412 | 0.0504 |
| 10 | -0.5 | 0.0512 | 0.0516 | 0.0484 | 0.0532 | 0.0512 | 0.0568 |
| 30 | | 0.0380 | 0.0488 | 0.0332 | 0.0502 | 0.0374 | 0.0484 |
| 10 | -0.9 | 0.0516 | 0.0524 | 0.0504 | 0.0588 | 0.0498 | 0.0556 |
| 30 | | 0.0364 | 0.0460 | 0.0402 | 0.0510 | 0.0364 | 0.0524 |

Tables 6-11 show the empirical power under different alternative hypotheses, $(H_A : \gamma_1 \neq \gamma_2 \neq \gamma_3)$ for comparing three groups. In general, Tables 6 – 11 show that using a RSS scheme results in more powerful testing procedure, than using SRS method for all given correlation coefficients. Also, when using RSS in the regression model to compare the groups' effects, the power increases as the mean difference (d), sample size, and correlation coefficient increases.

Table 6: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.1$ & $d2=0.3$).

| $1 - \beta$ | | | | | | | | |
|--|-----|-------------|---------|--------|---------|--------|---------|--------|
| H_A : at least one treatment γ not equal zero | | | | | | | | |
| d | r | ρ | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| | | | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.1/0.3 | 10 | 0.3 | 0.0757 | 0.0706 | 0.0780 | 0.0680 | 0.0834 | 0.0686 |
| | 30 | | 0.0762 | 0.0904 | 0.1154 | 0.0110 | 0.1586 | 0.1154 |
| | 10 | 0.5 | 0.0700 | 0.0700 | 0.0750 | 0.0740 | 0.0928 | 0.0680 |
| | 30 | | 0.0978 | 0.0970 | 0.1360 | 0.1068 | 0.1902 | 0.1420 |
| | 10 | 0.9 | 0.1500 | 0.0800 | 0.1670 | 0.1280 | 0.2564 | 0.1746 |
| | 30 | | 0.3290 | 0.2768 | 0.4828 | 0.3398 | 0.6772 | 0.4242 |
| | 10 | -0.3 | 0.0846 | 0.0506 | 0.0940 | 0.0610 | 0.1078 | 0.0612 |
| | 30 | | 0.0640 | 0.0632 | 0.0784 | 0.0634 | 0.0876 | 0.0694 |
| | 10 | -0.5 | 0.0778 | 0.0810 | 0.1010 | 0.0986 | 0.1346 | 0.1014 |
| | 30 | | 0.0648 | 0.0614 | 0.0790 | 0.0560 | 0.0894 | 0.0736 |
| | 10 | -0.9 | 0.0772 | 0.0758 | 0.0950 | 0.0798 | 0.1220 | 0.1054 |
| | 30 | | 0.0614 | 0.0550 | 0.0638 | 0.0574 | 0.0814 | 0.0616 |

Table 7: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.5$ & $d2=0.3$).

| $1 - \beta$ | | | | | | | | |
|--|-----|-------------|---------|--------|---------|--------|---------|--------|
| H_A : at least one treatment γ not equal zero | | | | | | | | |
| d | r | ρ | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| | | | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.5/0.3 | 10 | 0.3 | 0.0908 | 0.0828 | 0.1130 | 0.0900 | 0.1600 | 0.1104 |
| | 30 | | 0.1880 | 0.1492 | 0.2900 | 0.2072 | 0.4064 | 0.2524 |
| | 10 | 0.5 | 0.1000 | 0.0845 | 0.1430 | 0.1020 | 0.1844 | 0.1264 |
| | 30 | | 0.2196 | 0.1936 | 0.3360 | 0.2464 | 0.4952 | 0.3024 |
| | 10 | 0.9 | 0.2842 | 0.2312 | 0.4310 | 0.2920 | 0.6056 | 0.3744 |
| | 30 | | 0.7516 | 0.6260 | 0.9248 | 0.7600 | 0.9844 | 0.8508 |
| | 10 | -0.3 | 0.0876 | 0.0740 | 0.1068 | 0.0920 | 0.1660 | 0.0968 |
| | 30 | | 0.1684 | 0.1440 | 0.2384 | 0.1948 | 0.3564 | 0.2088 |
| | 10 | -0.5 | 0.0876 | 0.0844 | 0.1156 | 0.0832 | 0.1396 | 0.0916 |
| | 30 | | 0.1360 | 0.1288 | 0.2193 | 0.1705 | 0.3160 | 0.1912 |
| | 10 | -0.9 | 0.0725 | 0.0712 | 0.0992 | 0.0668 | 0.1096 | 0.0856 |
| | 30 | | 0.1060 | 0.0944 | 0.1616 | 0.1308 | 0.2208 | 0.1496 |

Table 8: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.8$ & $d2=0.3$).

| $1 - \beta$ | | | | | | | | |
|--|-----|-------------|---------|--------|---------|--------|---------|--------|
| H_A : at least one treatment γ not equal zero | | | | | | | | |
| d | r | ρ | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| | | | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.8/0.3 | 10 | 0.3 | 0.1686 | 0.1432 | 0.2550 | 0.1690 | 0.3672 | 0.2166 |
| | 30 | | 0.4626 | 0.3642 | 0.6746 | 0.4848 | 0.8362 | 0.5740 |
| | 10 | 0.5 | 0.1985 | 0.1594 | 0.3050 | 0.2010 | 0.4262 | 0.2554 |
| | 30 | | 0.5607 | 0.4409 | 0.7694 | 0.5520 | 0.9154 | 0.6688 |
| | 10 | 0.9 | 0.6678 | 0.5532 | 0.8760 | 0.6590 | 0.9610 | 0.7780 |
| | 30 | | 0.9934 | 0.9658 | 0.9998 | 0.9932 | 0.9999 | 0.9992 |
| | 10 | -0.3 | 0.1554 | 0.1328 | 0.2334 | 0.1652 | 0.3324 | 0.1730 |
| | 30 | | 0.3980 | 0.3070 | 0.5900 | 0.4102 | 0.7714 | 0.5112 |
| | 10 | -0.5 | 0.1486 | 0.1115 | 0.2097 | 0.1531 | 0.3042 | 0.1600 |
| | 30 | | 0.3401 | 0.2829 | 0.5274 | 0.3597 | 0.7025 | 0.4445 |
| | 10 | -0.9 | 0.1196 | 0.0954 | 0.1622 | 0.1150 | 0.2222 | 0.1202 |
| | 30 | | 0.2424 | 0.2100 | 0.3882 | 0.2682 | 0.5366 | 0.3204 |

Table 9: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.3$ & $d2=1.0$).

| | | | $1 - \beta$ | | | | | |
|---------|-----|-------------|--|--------|---------|--------|---------|--------|
| | | | H_A : at least one treatment γ not equal zero | | | | | |
| | | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| d | r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.3/1.0 | 10 | 0.3 | 0.2746 | 0.2054 | 0.3994 | 0.2654 | 0.5568 | 0.3434 |
| | 30 | | 0.6797 | 0.5803 | 0.8820 | 0.6986 | 0.9714 | 0.7903 |
| | 10 | 0.5 | 0.3034 | 0.2340 | 0.4700 | 0.3180 | 0.6420 | 0.4074 |
| | 30 | | 0.7920 | 0.6371 | 0.9406 | 0.7811 | 0.9894 | 0.8726 |
| | 10 | 0.9 | 0.8606 | 0.7517 | 0.9771 | 0.8886 | 0.9974 | 0.9388 |
| | 30 | | 1 | 0.9991 | 1 | 0.9997 | 1 | 1 |
| | 10 | -0.3 | 0.2731 | 0.1917 | 0.4277 | 0.2737 | 0.5737 | 0.3340 |
| | 30 | | 0.7048 | 0.5497 | 0.8982 | 0.6951 | 0.9717 | 0.7903 |
| | 10 | -0.5 | 0.3440 | 0.2668 | 0.5034 | 0.3280 | 0.6777 | 0.3886 |
| | 30 | | 0.7866 | 0.6460 | 0.9411 | 0.7831 | 0.9883 | 0.8731 |
| | 10 | -0.9 | 0.8837 | 0.7674 | 0.9743 | 0.8891 | 0.9980 | 0.9354 |
| | 30 | | 1 | 0.9980 | 1 | 1 | 1 | 1 |

Table 10: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.5$ & $d2=1.0$).

| | | | $1 - \beta$ | | | | | |
|---------|-----|-------------|--|--------|---------|--------|---------|--------|
| | | | H_A : at least one treatment γ not equal zero | | | | | |
| | | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| d | r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.5/1.0 | 10 | 0.3 | 0.2483 | 0.2048 | 0.3906 | 0.2628 | 0.5317 | 0.3080 |
| | 30 | | 0.6666 | 0.5283 | 0.8640 | 0.6731 | 0.9665 | 0.7803 |
| | 10 | 0.5 | 0.2963 | 0.2377 | 0.4466 | 0.3120 | 0.6081 | 0.3742 |
| | 30 | | 0.7514 | 0.6208 | 0.9160 | 0.7628 | 0.9837 | 0.8468 |
| | 10 | 0.9 | 0.8514 | 0.7406 | 0.9666 | 0.8500 | 0.9960 | 0.9297 |
| | 30 | | 0.9997 | 0.9985 | 1 | 0.9991 | 1 | 1 |
| | 10 | -0.3 | 0.2717 | 0.1983 | 0.4134 | 0.2517 | 0.5528 | 0.3240 |
| | 30 | | 0.6634 | 0.5257 | 0.8762 | 0.6731 | 0.9617 | 0.7808 |
| | 10 | -0.5 | 0.3228 | 0.2468 | 0.4706 | 0.3068 | 0.6488 | 0.3837 |
| | 30 | | 0.7651 | 0.6191 | 0.9291 | 0.7520 | 0.9871 | 0.8503 |
| | 10 | -0.9 | 0.8648 | 0.7354 | 0.9745 | 0.8588 | 0.9965 | 0.9297 |
| | 30 | | 1 | 0.9980 | 1 | 1 | 1 | 1 |

Table 11: Empirical Power For Testing Three Interventional Groups With Mean Different ($d1=0.8$ & $d2=1.0$).

| | | | $1 - \beta$ | | | | | |
|---------|-----|-------------|--|--------|---------|--------|---------|--------|
| | | | H_A : at least one treatment γ not equal zero | | | | | |
| | | | $m = 3$ | | $m = 4$ | | $m = 5$ | |
| d | r | ρ | RSS | SRS | RSS | SRS | RSS | SRS |
| 0.8/1.0 | 10 | 0.3 | 0.2776 | 0.2206 | 0.4260 | 0.2974 | 0.5902 | 0.3574 |
| | 30 | | 0.7271 | 0.5733 | 0.9094 | 0.7356 | 0.9802 | 0.8148 |
| | 10 | 0.5 | 0.3268 | 0.2640 | 0.4932 | 0.3356 | 0.6694 | 0.4156 |
| | 30 | | 0.8021 | 0.6805 | 0.9562 | 0.8042 | 0.9906 | 0.8854 |
| | 10 | 0.9 | 0.8896 | 0.7926 | 0.9836 | 0.8938 | 0.9978 | 0.9546 |
| | 30 | | 1 | 0.9989 | 1 | 0.9998 | 1 | 1 |
| | 10 | -0.3 | 0.2938 | 0.2216 | 0.4362 | 0.2844 | 0.6100 | 0.3582 |
| | 30 | | 0.7305 | 0.5855 | 0.9098 | 0.7226 | 0.9804 | 0.8256 |
| | 10 | -0.5 | 0.3548 | 0.2664 | 0.5432 | 0.3310 | 0.6966 | 0.4270 |
| | 30 | | 0.8181 | 0.6707 | 0.9574 | 0.8138 | 0.9930 | 0.8982 |
| | 10 | -0.9 | 0.8984 | 0.7780 | 0.9816 | 0.8956 | 0.9984 | 0.9518 |
| | 30 | | 0.9998 | 0.9994 | 1 | 1 | 1 | 1 |

5.3. Final Remarks of the Results

The results of the simulation study indicate that the modified RSS scheme under the multiple regression approach in an ANCOVA model produced a probability of a type I error close to the nominal value ($\alpha = 0.05$), similar to using SRS. However, results show that the nominal value (α) for the modified RSS is smaller than the SRS for different sample sizes and different correlation coefficients, as seen in Table 1 and Table 5. Furthermore, the results in Tables 2–4 and Tables 6–11 also indicate that the empirical power analysis ($1 - \beta$), based on the RSS scheme, has substantially improved compared to the SRS method for both situations of mean differences comparison between two and three groups. Also, by referring to Figures 1 and 2 in Appendix A, using RSS improves the power of testing for comparison effects in the event of using ranked auxiliary covariate (X) compared to using non-ranked covariate in a regression model approach. In addition, if the correlation coefficient between X and Y is negative, then the results testing tend to be less powerful than when it is positive for two experimental groups. However, using the proposed RSS is more powerful method than SRS.

Chapter 6

BOOSTING HAPPINESS AND BUTTRESSING RESILIENCE: UPLIFTING INTERVENTION DESIGNED BY USING RSS APPROACH

6.1. Objectives of the Chapter

In this chapter, we will provide an application where the modified *Ranked Set Sampling* (RSS) protocol was efficiently used as a cost-effective alternative to SRS as applied to a human population. The purpose of this chapter is to discuss a scenario where the RSS scheme can be used in an experimental study to compare two experimental groups. Subjects were selected for the experiment based on one available and less expensive to obtain auxiliary covariate. The following experimental study was conducted with the purpose of obtaining a better understanding of the pathways by which positive experiences (i.e., goal completion) contribute to higher levels of happiness, well-being, and life satisfaction. Undergraduate students at Georgia Southern University were the study population for this experiment. Dr. Jeff Klibert, and other scholars, conducted this research in the Department of Psychology, in collaboration with Dr. Hani Samawi, Dr. Haresh Rochani, Yisong Huang, and Rajai Jabra from the College of Public Health, Department of Biostatistics. It was expected that using RSS would increase the power of the statistical analysis of comparing two experimental condition groups, hence reduce the sample size required for this experimental study.

6.2. Introduction

The pursuit and attainment of happiness is of great interest to millions of people, however, there are few scientific theories that offer a process by which people pursue positive psychological states, like happiness (Bergsma, 2008; John C. Norcross et al., 2000). Every day, humans engage in a number of behaviors that have the potential to enhance their overall sense of happiness and life satisfaction (Tugade & Fredrickson, 2004). However, despite engagement in these behaviors, adolescents and emerging adults are more likely to report higher levels of negative emotions when compared to positive emotions (Santrock, 2012). Research indicates that a connection between positive experiences and greater levels of positive outcomes depends upon how individuals react to the positive experiences (Bryant & Veroff, 2007). For instance, individuals can engage in emotional uplifting strategies or dampening strategies in response to a positive experience. Emotional uplifting strategies are defined as behaviors that prolong or increase positive emotions, such as smiling or recalling a positive experience, whereas emotional dampening strategies are defined as behaviors that suppress positive reactions to positive experiences such as self-criticizing, and fault finding (Quoidbach et al., 2010).

To date, research has generated correlational evidence that suggests uplifting and dampening strategies are important in explaining the connection between positive experiences and positive outcomes (Quoidbach et al., 2010). However, correlational research cannot infer causation. As such, the current study, proposed by the department of Psychology and the College of Public Health at Georgia Southern University, has aimed to examine the relation between a gratitude-based emotional uplifting strategy, and key positive psychological states/resources such as resilience, efficacy, and happiness. Dr. Jeff Klibert, an Associate Professor of Psychology, coordinated and supervised all aspects of the experiment including administering the participant

tasks and procedures. Whereas, Dr. Hani Samawi, a Professor of Biostatistics, with other colleagues, implemented the protocol of selecting subjects for the intervention experimental study, by using the modified RR suggested by this dissertation, and randomly assigning experimental conditions to students.

6.3. The Purpose of the Study

The impetus of the study was multifaceted. The first purpose was to examine the impact of a gratitude-based, emotional uplifting strategy on positive emotion across the duration of the experiment. Individuals were selected for the experimental study based on their ranked well-being scores and then randomly placed in an uplifting intervention group and in a control group. It was hypothesized that the intervention group status would interact with time to explain fluctuations in positive mood. The second purpose of the study was to examine if the intervention placement would impact rates by which participants reported on indices of resilience, efficacy, and happiness. It was expected that participants placed in the gratitude-based, emotional uplifting intervention would report greater levels of resilience, efficacy, and happiness compared to individuals placed in the control group. The third goal of the study was to examine the effect of the implementation of the modified RR in subject selection to improve the analysis of means comparisons of the experimental groups.

6.4. Description of the Subjects

Approximately 250 undergraduate students at Georgia Southern University were recruited for the experiment study in two phases. All participants over the age of 18, and enrolled in a Psychology course, were eligible for the intervention experiment. In the first phase of data

collection, the 250 recruited undergraduate students completed a short series of surveys, and were asked if they would like to participate in the second phase of the study. Participants who expressed interest in completing the second phase of the study, served as the participant pool for phase II. Selection of participants for phase II was determined by variation in well-being score ranks, which was the baseline auxiliary covariate. From a list of 250 coded students and their well-being scores from phase I, only 183 students reported interest to be selected for phase II of the experiment. Considering there are only two experimental groups in the study, the gratitude-based, emotional uplifting intervention group and the control group, we only selected 60 students during phase II. To select those 60 students (30 in each group) we selected randomly 180 students out of the pool of 183 students from phase I.

6.5. Recruitment

- **Phase I**

In the first phase of the study, approximately 250 participants were recruited through SONA, an organizational system that allows participants to sign up for research studies via the internet. The SONA system is owned and operated by the Psychology Department at Georgia Southern University. Students, who chose to participate in the research study, registered their names, course number, and the professor of the course through the SONA system. Recruited participants were asked to complete a survey, which included: the Demographic Questionnaire and the Ryff Psychological Well-Being Scale-Short Form (RPWS-SF). The RPWS-SF is an 18-item measure of psychological well-being, defined as a state of contentment marked by self-acceptance, positive relations with others, autonomy, environmental mastery, purpose in life, and personal growth (Ryff, 1989). Participants rate their agreement on each item on a 6-point Likert

scale, with responses ranging from “strongly agree” to “strongly disagree.” Total scores are summative and range from 18 to 108, with higher scores indicating greater levels psychological well-being. After the questionnaires were completed, the participants were directed to an invitation page, where they were asked to consider participating in phase II of the study. If interested in phase II, the participants were asked to provide their contact information. To encourage interest in participating in the second phase of the study, the researcher held a drawing for one fifty-dollar Amazon gift card.

- **Phase II**

Students who completed phase I of the study and expressed interest to continue participating the experiment, were added to our selection pool for participant recruitment in phase II. The selection process of students for phase II intervention by using the modified RSS protocol was implemented by the following:

- 1- We randomly selected 180 students from the pool of 183 students, who had shown interest to continue the intervention.
- 2- We randomly shuffled the subjects and randomly selected $L = 2$ sets, each set contained $m = 9$ students.
- 3- With the selected a set of 9 students in both groups. We ranked the first 3 students based on well-being scores and selected the student, with only the minimum well-being score to be included in the second phase, and randomly assigned them to one of the groups. Unselected students were discarded.
- 4- We repeated step 3 on the next 3 students on each set, but selected only students with the moderate well-being score to be included in phase II of the experiment.

- 5- We repeated step 3 again on the next 3 students on each group, but selected only students with the maximum well-being score to be included in phase II of the experiment.
- 6- From step 3-5 we had two groups of three students selected based on the ranking of phase I well-being scores, which called the first cycle of RSS of size 3, and randomly assigned into the two intervention groups.
- 7- We repeated steps 3-6 nine times to have a RSS of 30 students in each group.

Students who had been selected to participate were contacted to complete the second phase of the study with one of the clinical researchers. Participation in the second phase of the study completed a number of experimental tasks in the primary researcher's lab. Upon entering the lab, all participants were asked to place their cell-phones and other belongings in a small storage space for the duration of the study. The full process took approximately 75 minutes to be completed.

6.6. Intervention Groups

Selected participants for the gratitude-based emotional uplifting intervention group and the control group were asked to complete a number of tasks for the experiment. They were asked to complete one short questionnaire about their mood, to measure positive and negative emotions (Positive and Negative Affect Schedule [PANAS]). The PANAS is a brief, 20-item measure of positive and negative mood states. Respondents were asked to rate their current emotional state based upon 20 unique, emotional prompts. Participants were asked to rate their mood on a scale between 1 and 100. After the participants completed the first set, they were asked to complete a narrative-based, writing task. The writing task asked participants to reflect on their personal experience with goal completion/achievement. The purpose of the task was to elicit positive

memories and emotions associated with a challenging, yet rewarding, experiences. The duration of the writing task was approximately 10-15 minutes in length. After they completed the writing task, participants were asked to complete the mood and emotion questionnaire (PANAS) for the second time, as a manipulation check to determine if the writing task was effective in eliciting positive emotions (e.g., pride, interest, activity, etc.).

After completing the second set of emotion questionnaires, individuals assigned to the emotional uplifting intervention group were asked to extend positive emotions through a gratitude-based task. The purpose of the exercise was to help individuals mindfully extend and enhance positive emotions associated with a goal completion/achievement, whereas, the second group of participants in the control group, were not directed to complete an assigned task. Instead, they were asked to wait in a private room until the Research Assistant instructed them further. Completion of both tasks was expected to last approximately 20 minutes. After completing one of the assigned tasks, all participants were then asked to complete a series of self-report measures. These measures will include the PANAS, the Coping-Self-Efficacy Scale (a valid measure of coping self-efficacy), the CD-RISC (a valid measure of resilience), OHQ (a valid measure of happiness), and a demographic form. In total, it was expected that participants were able to complete the entire process in approximately 70-75 minutes.

6.7. Data Analysis

This experiment was designed to investigate the relations between positive experiences and contributing higher levels of happiness, well-being, and life satisfaction. The uplifting intervention strategy group was assessed by measuring the reported positive mood and compared with the control group. The hypothesis testing for this research is that “the increase in mood

would be experienced more greatly in participants who were selected into the intervention group compared to the control group.” In other words, the hypothesis testing can be reformed as the following,

$$H_0 : \gamma_{(intervention)} = \gamma_{(control)} = 0$$

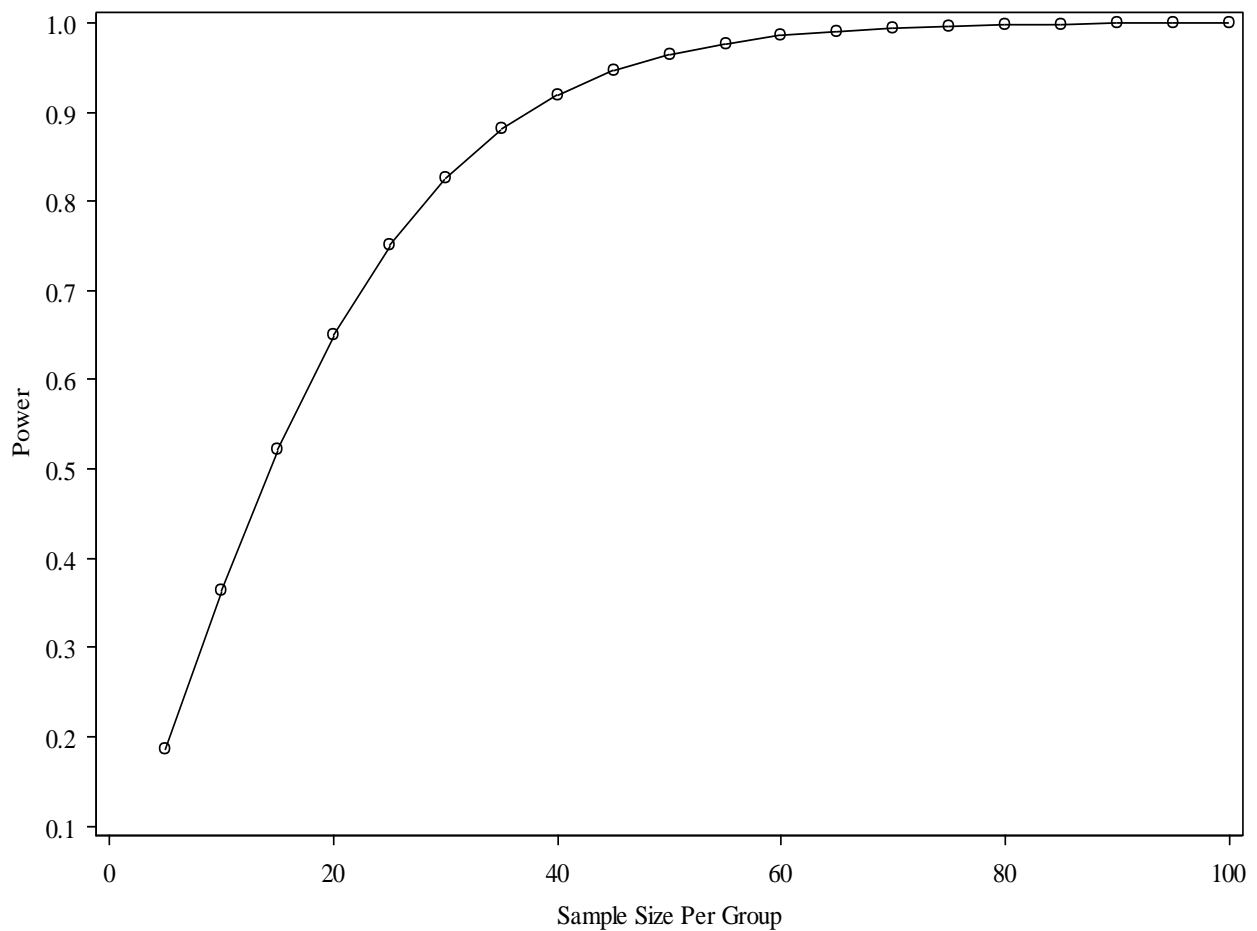
The comparison in the intervention groups was accomplished by using covariance analysis (ANCOVA) models. We are expecting a significant interaction term between positive mood (repeated measures) and groups (intervention vs. control). Out of the selected 60 students for the phase II intervention, 45 students have responded and completed the second set of questioners, 23 students in the uplifting group and 22 students in the control group. Descriptive analysis of the participants by intervention groups is given in Table 12.

Table 12: *Descriptive Characteristics By Intervention Groups: Uplifting Experimental Study.*

| <u>Characteristic</u> | <u>Uplifting Intervention</u> | | <u>Control Group</u> | |
|-----------------------|--------------------------------------|----------|-----------------------------|----------|
| | N | % | N | % |
| Age | | | | |
| 18-19 | 17 | 73.9 | 10 | 45.5 |
| 20-21 | 6 | 26.1 | 11 | 50 |
| 22+ | 0 | 0 | 1 | 4.5 |
| Race | | | | |
| African American | 6 | 26.1 | 5 | 22.7 |
| White | 16 | 69.6 | 14 | 63.6 |
| Hispanic | 1 | 4.3 | 1 | 4.6 |
| Others | 0 | 0 | 3 | 13.6 |
| Gender | | | | |
| Male | 8 | 34.8 | 5 | 22.7 |
| Female | 15 | 62.2 | 17 | 77.3 |

With a sample size of 45 subjects, the estimated power for the analysis of covariance is 0.73. As it is shown in Figure 1, the estimates for the statistical power of analysis for a sample size of 60, 30 students in each group, is more than 0.8. Therefore, RSS is an effective methodology for subject selection to improve the outcome means comparisons of intervention groups with small sample sizes.

Figure1: Power Analysis For Positive Mood Outcome For Uplifting Intervention By Using RSS Scheme.



To assess the experiment intervention, the mean differences of the final positive mood over the gratitude-based, emotional uplifting intervention group and the control group were compared. For testing the hypothesis, a multiple regression approach in an ANCOVA model was used to investigate positive mood contribution in both the uplifting intervention group and the control group, results are given in Table 13.

From the analysis in Table 13, we concluded that there were significant differences in final positive mood between the intervention and the control groups, $p\text{-value} = 0.0095$. However,

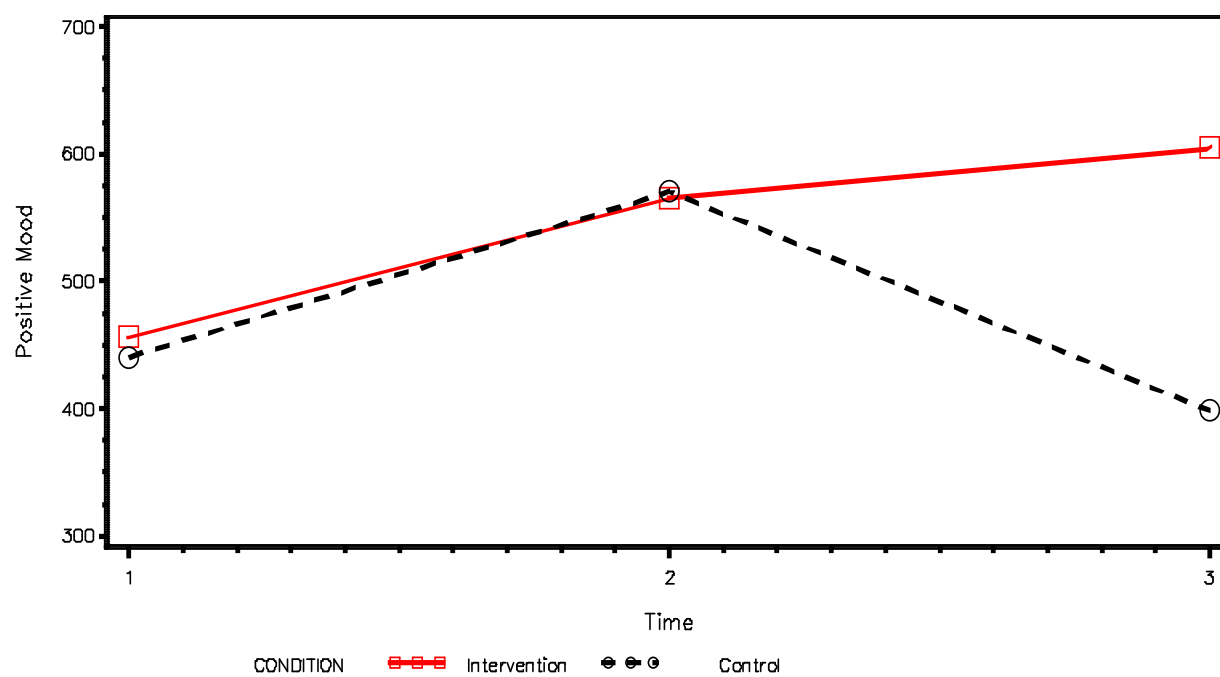
the characteristic covariates including age, marital status, race and academic class were not significant factors.

Table 13: *Analysis of Covariance in a Multiple Regression Model*

| Parameter | Estimate | Standard Error | t Value | Pr > t |
|-------------------------|-----------------|-----------------------|----------------|--------------------|
| Intercept | 724.8051914 | E 357.4163244 | 2.03 | 0.0489 |
| Condition 1 | 210.1576518 | E 77.2933222 | 2.72 | 0.0095 |
| Well Being Total | -3.9843118 | 4.3088167 | -0.92 | 0.3604 |

The positive mood outcome for the uplifting intervention group and the control group was assessed in three different periods, baseline, and midterm evaluation. Then at the end of the experiment evaluation, the mean different of positive mood outcome over the three periods evaluation has been given in Figure 2. It shows that the final mean positive mood outcome for students who participated in the uplifting intervention was 627; whereas, the final mean positive mood score for students participated the control group was only 390. These results indicate that the uplifting intervention leads to greater reports of positive mood and happiness. On the other hand, the power analysis suggested that the experimental study with a small sample size, i.e. 60 participants, was very efficient by adopting the modified RSS protocol to select subjects into either intervention or control groups.

Figure 2: *Uplifting Intervention and Control Final Positive Mood by Time.*



Chapter 7

Discussion and Future Work

7.1. Summary

Ranked Set Sampling (RSS) method was originally proposed by (McIntyre, 1952) to use in ecological and environmental studies, to estimate the mean of a population as an efficient and a cost-effectiveness alternative to the *Simple Random Sampling* (SRS), when the outcome of interest is difficult to obtain. In recent years, RSS has been applied to problems in ecological and environmental science, and has also been adopted in epidemiological applications. The work presented in this dissertation is an attempt to frame a modified approach of the RSS to be efficiently applied in experimental studies for the aim of comparing L groups.

RSS is beneficial, especially when the measurements of interest are expensive and/or not easy to obtain. Along with the experimental outcome, other information such as auxiliary covariates are highly correlated to the outcome of interest and can be used to estimate the population mean by reducing the required sample size. In addition to the intensive literature, as discussed in Chapter 2 and Chapter 3, this dissertation suggests that those baseline covariates can be used to improve the inference of intervention comparisons when the measures of interest outcome is difficult to obtain in terms of cost and time. As seen in Chapter 5, the simulation results show that the nominal value (α) for using the modified RSS scheme are smaller than the SRS for all illustrated correlations, mean differences, and sample sizes.

In Chapter 4, we proposed a modified RSS scheme to examine the treatment effects between two or more experimental groups, by using one available auxiliary covariate, which have known to be correlated with the variable of interest. Under the proposed protocol, subjects

can be selected for an experimental study based on the ranked auxiliary covariate and randomly allocated into the experimental groups. Therefore, the modified RSS suggested in this dissertation has shown that it is an effective approach to select more structured and representative samples based on the most available and cost efficient auxiliary covariates.

An ANCOVA model in the multiple regression approach was proposed for making inferences with continuous auxiliary covariate data. The Weighted Least Squares (WLS) estimator has been adopted to estimate the model's parameters when the observed values of variability differ over the predictor values of the ranked subjects, where the weights are determined using the variance of the order statistics. Therefore, the WLS estimators for the auxiliary covariate parameter $\hat{\beta}_{RSS}$ and the groups indicator variable's parameter $\hat{\gamma}_{RSS}$ have been shown in Chapter 4 are unbiased estimators for the population's estimators β and γ and have less variances, respectively. On the other hand, the modified RR is more efficient than using SRS.

The main strength of this modified sampling scheme is that it is a cost-effective method for some experimental studies. As shown in Chapter 6, we have illustrated the proposed modified RSS in an experimental study conducted by the department of Psychology and the College of Public Health at Georgia Southern University. The method of the modified RSS has been used to select subjects into the experimental study and randomly assigned to a gratitude-based emotional uplifting intervention group and a control group. The key of adopting the RR is that smaller sample size can be considered for the experiment, and highly estimated power for the analysis can be achieved.

The selection process was accomplished based on one baseline ranked auxiliary covariate, where experimental subjects could be easily and economically obtained. As it is

shown in Chapter 6 - Figure 1, the estimated power for a total sample size of 60, with 30 students in each group, is more than 0.8. Therefore, it can be concluded that the modified RSS is an effective methodology for subject selection to improve the outcome means comparisons of intervention groups with small sample sizes. However, the biggest limitation of the proposed sampling scheme is that it may not be valid in clinical trials, where it is considered unethical to discard fully measured units (Chen, 2015).

7.2. Future Work

Evidently, RSS has regained interest among researchers in sampling designs, however, there are several topics that can be pursued in future research. More researchers and statisticians should look into the applications of RSS in their respective fields where it promises efficiency at lower cost. Furthermore, we would like to investigate the use and the performance of the modified RR in other study designs, such as in categorical data.

The balance of cost and precision is a complex problem for any research study. RSS provides a methodology for incorporating additional information into the sampling framework. However, unlike other two-phase sampling methods which that use “extra” information, RSS can be accomplished by using direct ranking such as expert judgment, or indirect ranking based on auxiliary covariates, without losing precision of estimation. For example, Nahhas et al. (2002) proposed a cost model for the problem of estimation of bone mineral density (BMD) in a human population. Subjects for such a study are numerous, but measurement of BMD via dual x-ray absorptiometry on the selected subjects is expensive. Thus, they demonstrated that a substantial increment in efficiency could be achieved by RSS to minimize the number of subjects required for such a study without reducing the amount of reliable information. Although, this dissertation

has shown that it is possible to reduce the required sample size, and therefore reduce the cost and increase efficiency, we have not proposed a cost-effectiveness model to estimate the actual reduction in cost of adopting the modified RSS. When ranking is not direct, as in many types of fieldwork, the precision gained from the auxiliary covariates must be balanced against the costs. The costs analyses can be proposed in future research assessing the RSS using the total costs of sampling.

In his dissertation, we investigated using an auxiliary covariate that is available to rank a sample, and then only used the measurement of the variable of interest in parameter estimation. However, dealing with a topic of ranking with missing data problem appears to be lacking in the literature. Therefore, future research can be pursued to propose the RSS as a missing data problem and examine the use of the measurements from both the ranking variable and the variable of interest in parameter estimation.

References

- Barreto, M. C. M., & Barnett, V. (1999). *Best linear unbiased estimators for the simple linear regression model using ranked set sampling*.
- Bergsma, A. (2008). Do self-help books help? *Journal of Happiness Studies*, 9(3), 341-360. doi: 10.1007/s10902-006-9041-2
- Bryant, F. B., & Veroff, J. (2007). *Savoring : a new model of positive experience*: Mahwah, N.J. : Lawrence Erlbaum Associates, 2007.
- Chen, D.-G. (2015). Innovative statistical methods for public health data. 291-313.
- Chen, Z., Bai, Z., & Sinha, B. K. (2004). *Ranked set sampling : theory and applications*: New York : Springer, c2004.
- Chen, Z., Liu, J., Shen, L., & Wang, Y. G. (2008). General ranked set sampling for efficient treatment comparisons. *STATISTICA SINICA*, 18, 91-104.
- Chen, Z., & Shen, L. (2003). Two-layer ranked set sampling with concomitant variables. *Journal of Statistical Planning and Inference*, 115, 45-57. doi: 10.1016/S0378-3758(02)00145-3
- Culpepper, S. A., & Aguinis, H. (2011). Using analysis of covariance (ANCOVA) with fallible covariates. *Psychological Methods*, 16(2), 166-178. doi: 10.1037/a0023355
- Daniel, J. (2012). *Sampling Essentials: Practical Guidelines for Making Sampling Choices*. Thousand Oaks, CA: SAGE.
- Dell, T. R., & Clutter, J. L. (1972). Ranked Set Sampling Theory with Order Statistics Background, 545.
- Donner, A., & Zou, G. Y. (2007). Methods for the statistical analysis of binary data in split-mouth designs with baseline measurements. *Statistics In Medicine*, 26(18), 3476-3486.
- Dura, C., Drigă, I., & Niță, D. (2010). STATISTICAL LANDMARKS AND PRACTICAL ISSUES REGARDING THE USE OF SIMPLE RANDOM SAMPLING IN MARKET RESEARCHES. *Annals of the University of Petrosani Economics*, 10(2), 111.
- Egger, J., Graham, P. J., Carter, C. M., Gumley, D., & Soothill, J. F. (1985). Controlled trial of oligoantigenic treatment in the hyperkinetic syndrome. *Lancet*.
- English, R. A., Lebovitz, Y., & Giffin, R. B. (2010). *Transforming clinical research in the United States : challenges and opportunities : workshop summary / Rebecca A. English, Yeanwoo Lebovitz, and Robert B. Giffin, rapporteurs ; Forum on Drug Discovery, Development, and Translation, Board on Health Sciences Policy, Institute of Medicine of the National Academies*: Washington, D.C. : National Academies Press, c2010.

- Evans, M. J. (1967). *Application of ranked set sampling to regeneration, Surveys in areas direct-seeded to long leaf pine*. (Master), Louisiana state University, Baton Rouge, Louisiana.
- Graybill, F. A. (1961). *An Introduction to Linear Statistical Model* (Vol. 1). New York.
- Halls, L. S., & Dell, T. R. (1966). Trial of ranked set sampling for forage yields. *Forest Science*, 12, 22-26.
- Hauck, W. W., Anderson, S., & Marcus, S. M. (1998). Should we adjust for covariates in nonlinear regression analyses of randomized trials? (Vol. 19, pp. 249-256).
- John C. Norcross, John W. Santrock, Linda F. Campbell, Thomas P. Smith, Robert Sommer, & Zuckerman, E. L. (2000). *Authoritative Guide to Self-Help Resources in Mental Health*. New York: Guilford Press.
- Kalsbeek, W., & Heiss, G. (2000). BUILDING BRIDGES BETWEEN POPULATIONS AND SAMPLES IN EPIDEMIOLOGICAL STUDIES. *Annual Review of Public Health*, 21(1), 147.
- Kandola, D., Banner, D., O'Keefe-McCarthy, S., & Jassal, D. (2014). Sampling Methods in Cardiovascular Nursing Research: An Overview. *Canadian Journal of Cardiovascular Nursing*, 24(3), 15-18 14p.
- Klockars, A. J., & Beretvas, S. N. (2001). Analysis of covariance and randomized block design with heterogeneous slopes. *Journal of Experimental Education*, 69(4), 393-410. doi: 10.1080/00220970109599494
- Koch, G. G., Tangen, C. M., Jung, J. W., & Amara, I. A. (1998). Issues for covariance analysis of dichotomous and ordered categorical data from randomized clinical trials and non-parametric strategies for addressing them. *Statistics In Medicine*, 17(15-16), 1863-1892.
- Lachin, J. M. (1988). Properties of simple randomization in clinical trials. *Controlled Clinical Trials*, 9(4), 312-326.
- Lam, K. F., Philip, L. H. Y., & Lee, C. F. (2002). Kernel method for the estimation of the distribution function and the mean with auxiliary information in ranked set sampling. *Environmetrics*, 13(4), 397-406.
- Lesaffre, E., & Senn, S. (2003). A note on non-parametric ANCOVA for covariate adjustment in randomized clinical trials. *Statistics In Medicine*, 22(23), 3583-3596.
- Linn, R. L., & Werts, C. E. (1971). Considerations for Studies of Test Bias, 1.
- MacEachern, S. N., Ozturk, O., Wolfe, D. A., & Stark, G. V. (2003). A new ranked set sample estimator of variance. *QUALITY CONTROL AND APPLIED STATISTICS*, 48, 189-192.

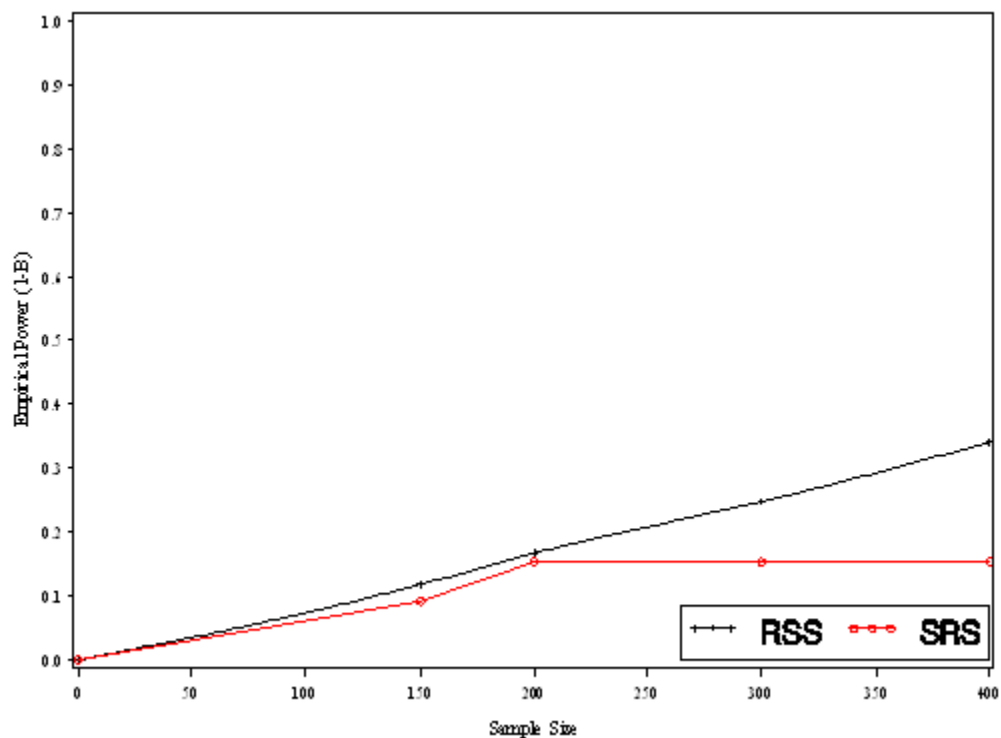
- McEntegart, D. (2003). The Pursuit of Balance Using Stratified and Dynamic Randomization Techniques: An Overview. *Drug Information Journal*, 37(3), 293.
- McIntyre, G. A. (1952). A METHOD FOR UNBIASED SELECTIVE SAMPLING USING RANKED SETS. *Australian Journal of Agricultural Research*, 3, 385-390. doi: 10.1071/ar9520385
- Muttalak, H. A. (1996). Estimation of parameters in a multiple regression model using ranked set sampling. *Information and Optimization Sciences*, 17, 521-533.
- Nahhas, R. W., Wolfe, D. A., & Chen, H. (2002). Ranked Set Sampling: Cost and Optimal Set Size, 964.
- Özdemir, Y. A., & Esin, A. A. (2007). PARAMETER ESTIMATION IN MULTIPLE LINEAR REGRESSION MODELS USING RANKED SET SAMPLING. *Communications Series AI Mathematics & Statistics*, 56(1), 7-20.
- Pocock, S. J., Assmann, S. E., Enos, L. E., & Kasten, L. E. (2002). Subgroup analysis, covariate adjustment and baseline comparisons in clinical trial reporting: current practice and problems (Vol. 21, pp. 2917-2930).
- Quoidbach, J., Berry, E. V., Hansenne, M., & Mikolajczak, M. (2010). Positive emotion regulation and well-being: Comparing the impact of eight savoring and dampening strategies. *Personality & Individual Differences*, 49(5), 368-373. doi: 10.1016/j.paid.2010.03.048
- Rao, C. R. (1965). *Linear statistical inference and its applications* [by] C. Radhakrishna Rao: New York, Wiley [1965].
- Rohatgi, V. K., & Saleh, E. (2001). *An introduction to probability theory and mathematical statistics* / V. K. Rohatgi: New York : Wiley, [1975].
- Ryff, C. D. (1989). Happiness Is Everything, or Is It? Explorations on the Meaning of Psychological Well-Being. *Journal of Personality & Social Psychology*, 57(6), 1069-1081.
- Samawi, & Ababneh, F. (2001). On regression analysis using ranked set sample. *Journal of Statistical Research (JSR)*, 35(2), 93-105.
- Samawi, H., & Pararai, M. (2010). *On the Optimality of Bivariate Ranked set Sample Design for Matched Pairs Sign Test*: Digital Commons@Georgia Southern.
- Samawi, H., & Vogel, R. (2015). On Ranked Set Sampling Variation and Its Applications to Public Health Research. In D.-G. D. Chen, Wilson, Jeffrey R (Ed.), *Innovative Statistical Methods for Public Health Data* (pp. 291-313).

- Samawi, H. M., & Abu-Dayyeh, W. (2002). On Regression Analysis with Random Regressors Using Ranked Samples. *INTERNATIONAL JOURNAL OF INFORMATION AND MANAGEMENT SCIENCES*, 13, 19-36.
- Samawi, H. M., & Al-Sagheer, O. A. M. (2001). On the Estimation of the Distribution Function Using Extreme and Median Ranked Set Sampling. *Biometrical Journal*, 43(3), 357.
- Samawi, H. M., & Muttalak, H. A. (1996). Estimation of Ratio Using Rank Set Sampling. *Biometrical Journal*, 38(6), 753-764. doi: 10.1002/bimj.4710380616
- Sandelowski, M. (2000). Focus on research methods. Combining qualitative and quantitative sampling, data collection, and analysis techniques in mixed-method studies. *Research in Nursing & Health*, 23(3), 246-255.
- Santrock, J. W. (2012). *Adolescence : an introduction*: Dubuque, Iowa : W.C. Brown, ©2012.14th ed.
- Senn, S. J. (1989). Covariate imbalance and random allocation in clinical trials. *Statistics In Medicine*, 8(4), 467-475.
- Sinha, A. K., Patil, G. P., & Taille, C. (1993). Relative precision of ranked set sampling: a comparison with the regression estimator. *Environmetrics*, 4(4), 399.
- Soekoe, M., van der Bank, F. H., & Smit, N. J. (2013). Determining the most suitable method of otolith preparation for estimating the age of tigerfish, *Hydrocynus vittatus* in the Pongolapoort Dam, South Africa. *African Zoology (Zoological Society of Southern Africa)*, 48(1), 187.
- Stokes. (1977). Ranked set sampling with concomitant variables. *Communications in Statistics: Theory & Methods*, 6(12), 1207.
- Stokes, S. L. (1980). Estimation of Variance Using Judgment Ordered Ranked Set Samples. *Biometrics*(1), 35. doi: 10.2307/2530493
- Stokes, S. L., & Sager, T. W. (1988). Characterization of a Ranked-Set Sample with Application to Estimating Distribution Functions, 374.
- Takahasi, K., & Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering. *Annals of the Institute of Statistical Mathematics*, 20(1), 1.
- Tsiatis, A. A., Davidian, M., Zhang, M., & Lu, X. (2008). Covariate adjustment for two-sample treatment comparisons in randomized clinical trials: a principled yet flexible approach. *Statistics In Medicine*, 27(23), 4658-4677.

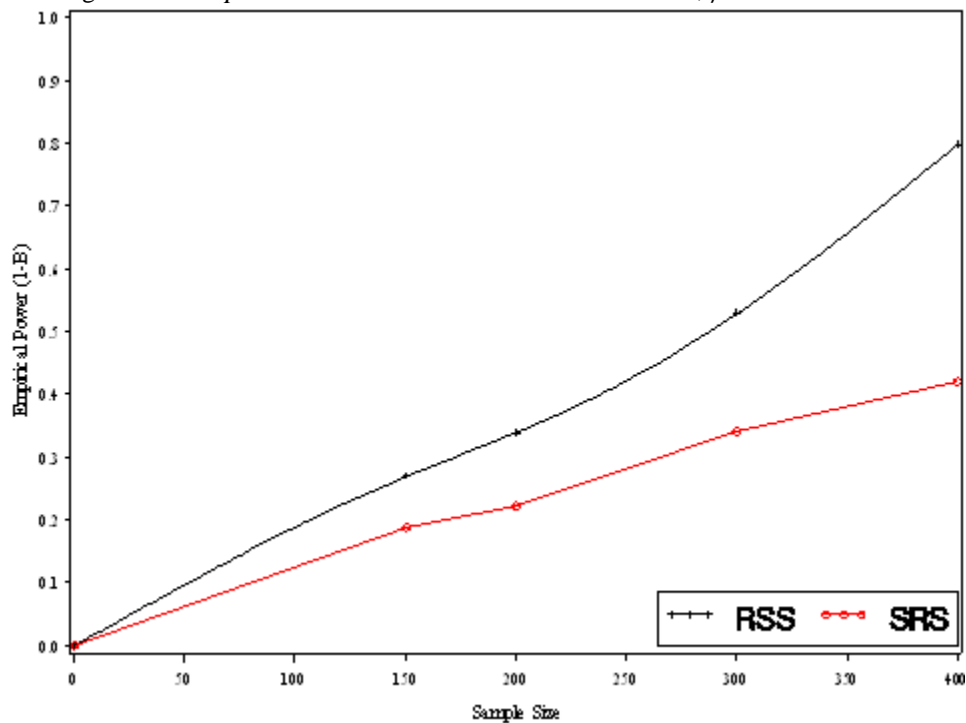
- Tugade, M. M., & Fredrickson, B. L. (2004). Resilient Individuals Use Positive Emotions to Bounce Back From Negative Emotional Experiences. *Journal of Personality and Social Psychology*, 86(2), 320-333. doi: 10.1037/0022-3514.86.2.320
- Zhang, Tsiatis, A. A., & Davidian, M. (2008). Improving Efficiency of Inferences in Randomized Clinical Trials Using Auxiliary Covariates. *Biometrics*, 64(3), 707-715.

APPENDIX A: Empirical Power Plots

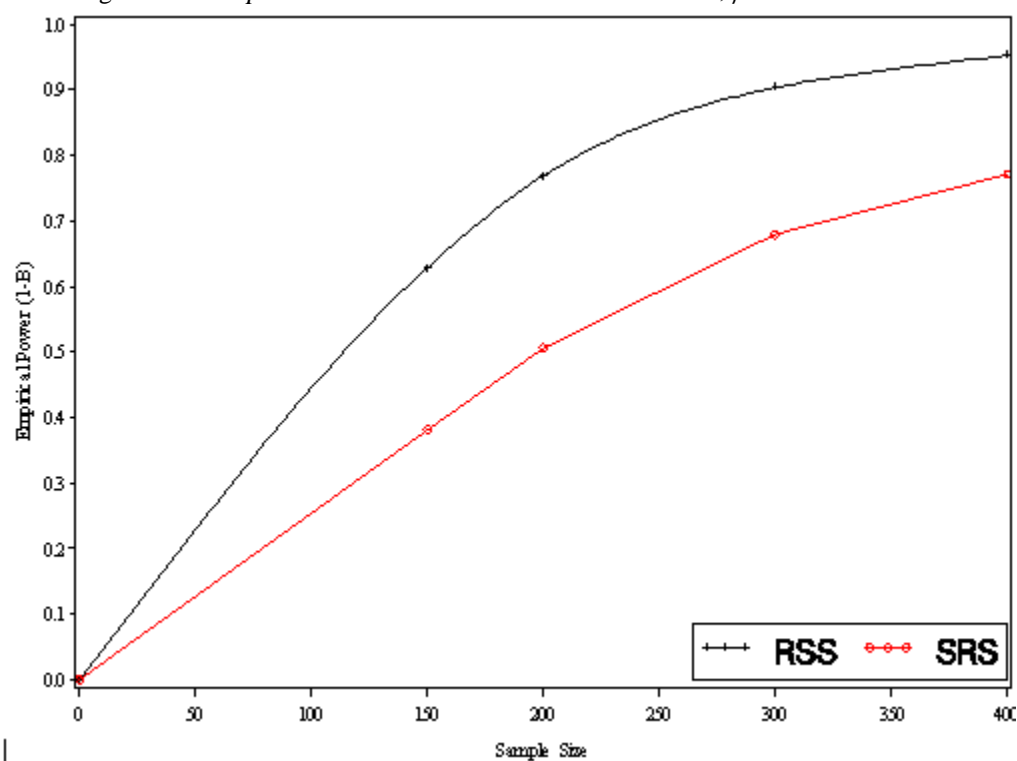
Figures 1.a. *Empirical Power For Positive Correlations, $\rho=0.3$ and $d=0.3$.*



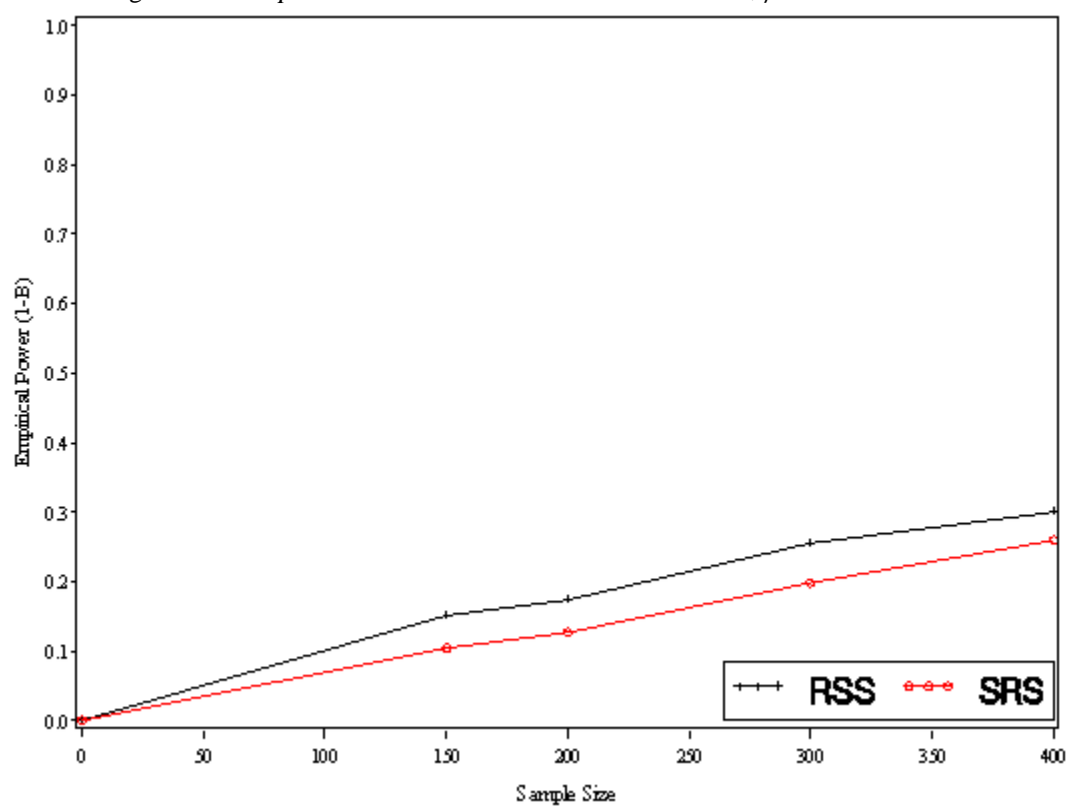
Figures 1.b *Empirical Power For Positive Correlations, $\rho=0.3$ and $d=0.5$.*



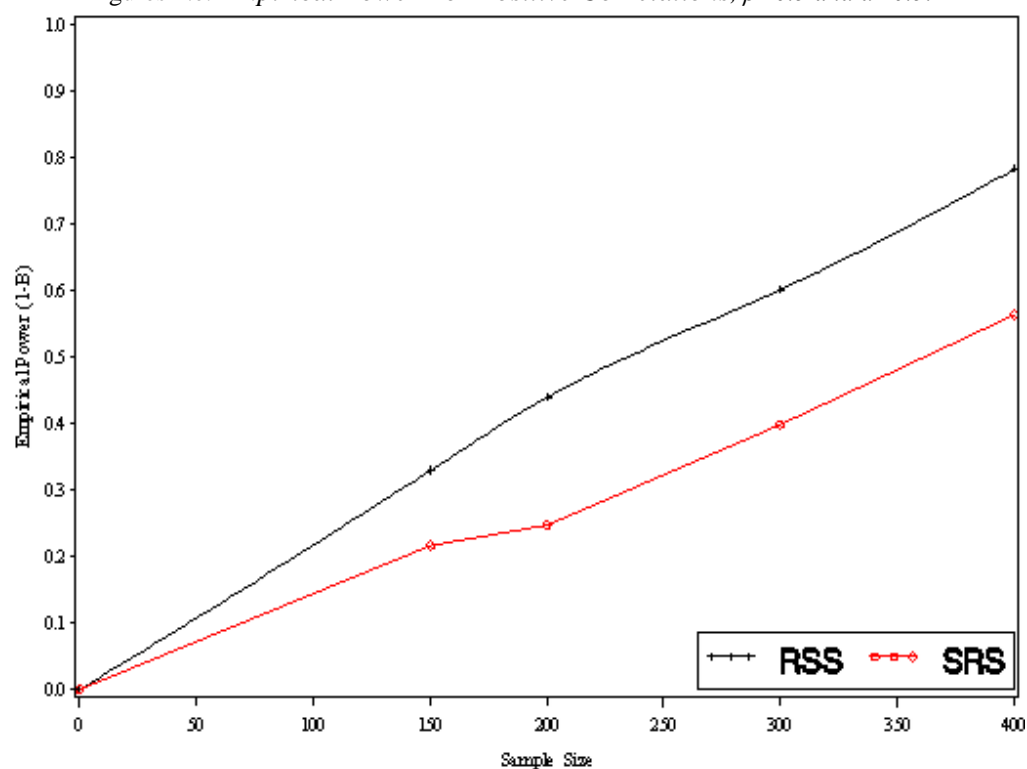
Figures 1.c. Empirical Power For Positive Correlations, $\rho=0.3$ and $d=0.8$.



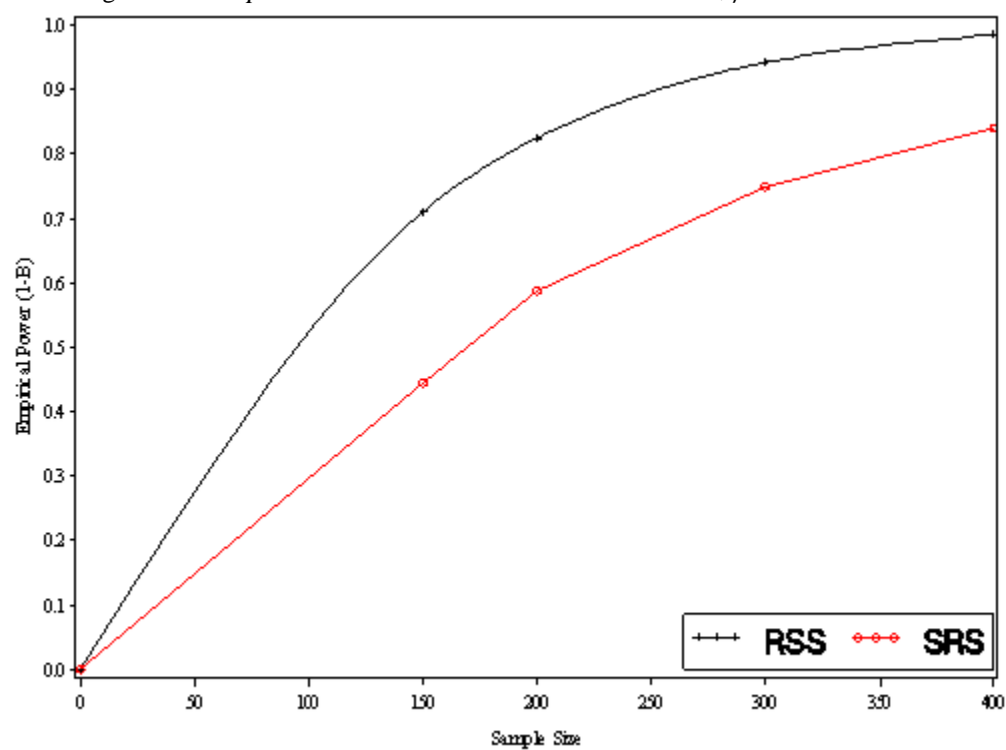
Figures 1.d. Empirical Power For Positive Correlations, $\rho=0.5$ and $d=0.3$.



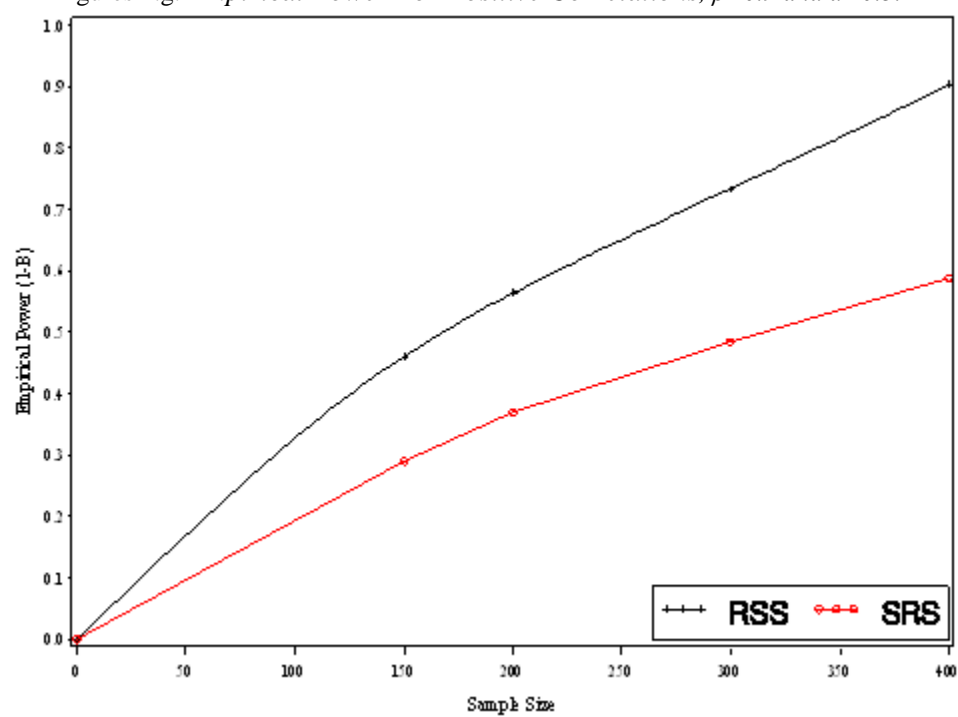
Figures 1.e. *Empirical Power For Positive Correlations, $\rho=0.5$ and $d=0.5$.*



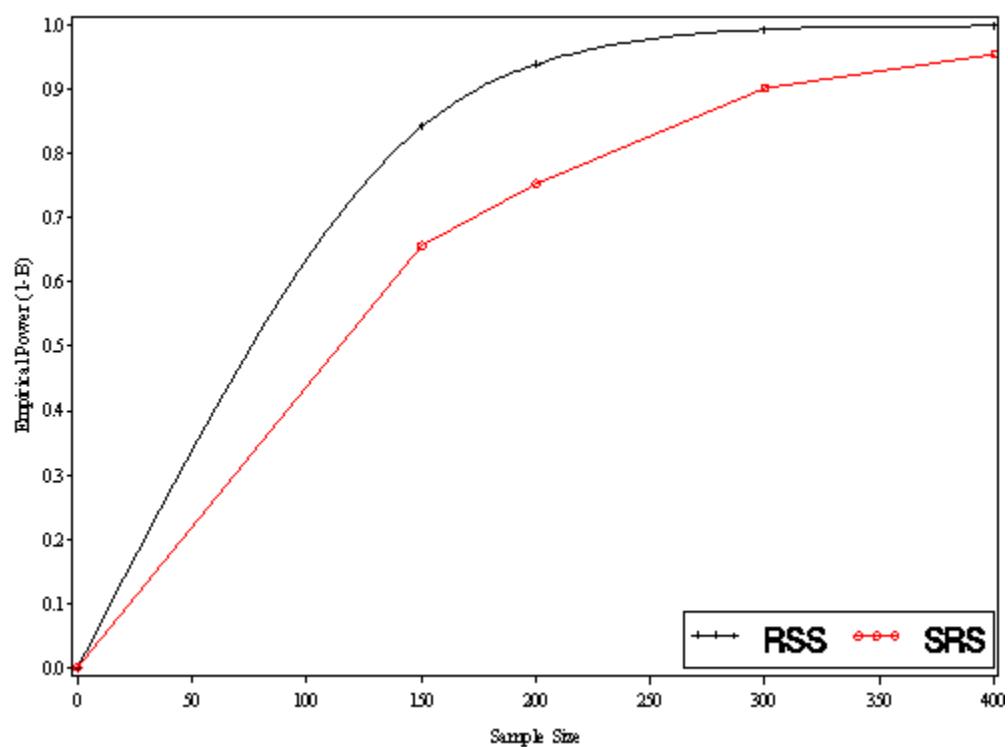
Figures 1.f. *Empirical Power For Positive Correlations, $\rho=0.5$ and $d=0.8$.*



Figures 1.g. *Empirical Power For Positive Correlations, $\rho=0.9$ and $d=0.3$.*



Figures 1.h. *Empirical Power For Positive Correlations, $\rho=0.9$ and $d=0.5$.*



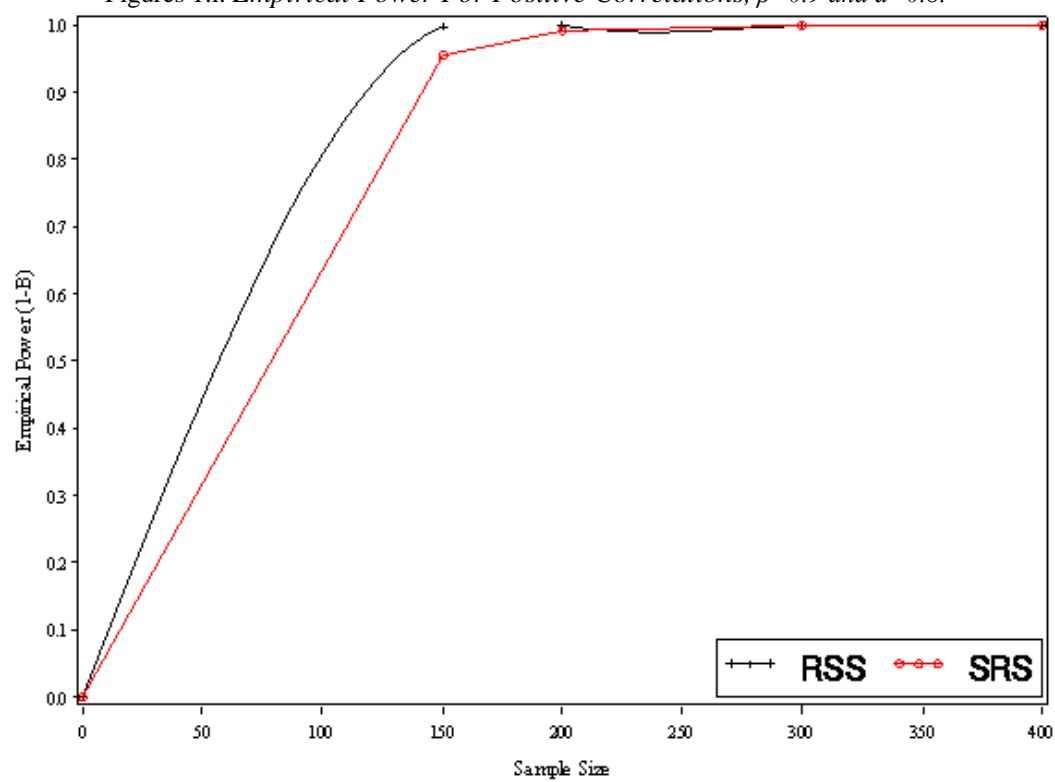
Figures 1.i. *Empirical Power For Positive Correlations, $\rho=0.9$ and $d=0.8$.*

Figure 2.a. Empirical Power For Negative Correlations, $\rho = -0.3$ and $d = 0.3$.

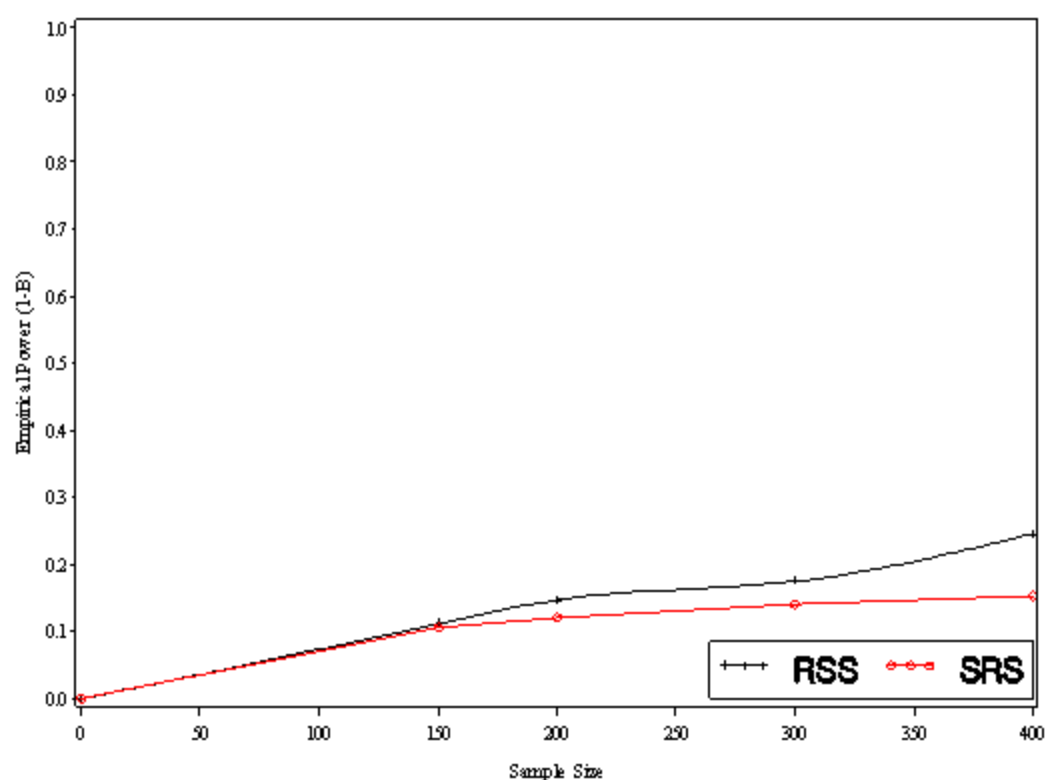


Figure 2.b. Empirical Power For Negative Correlations, $\rho = -0.3$ and $d = 0.5$.

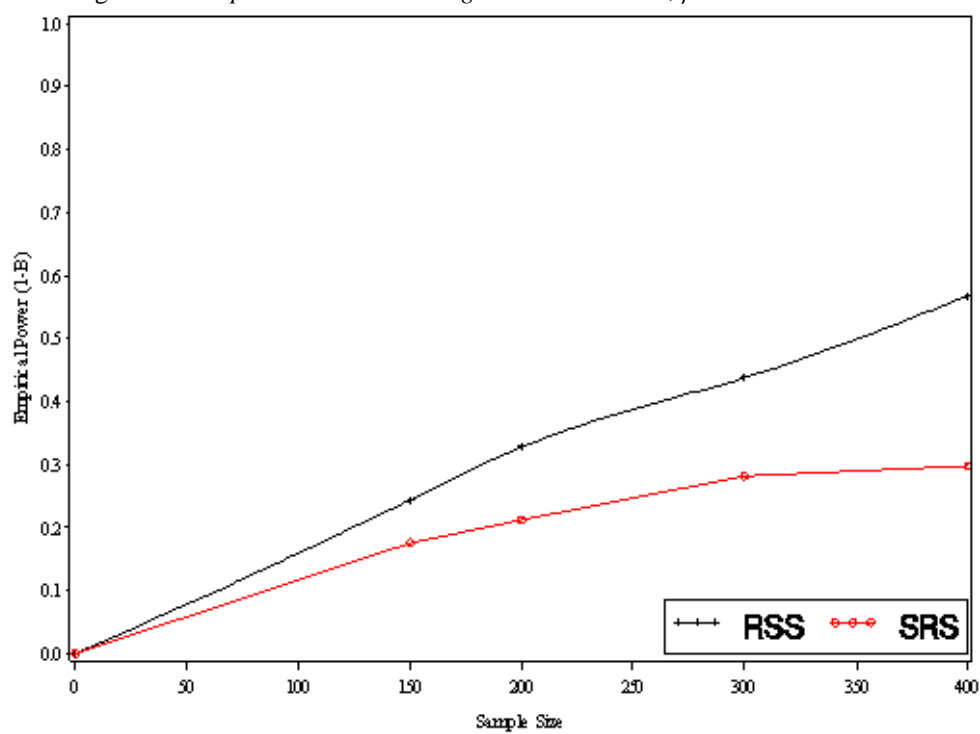


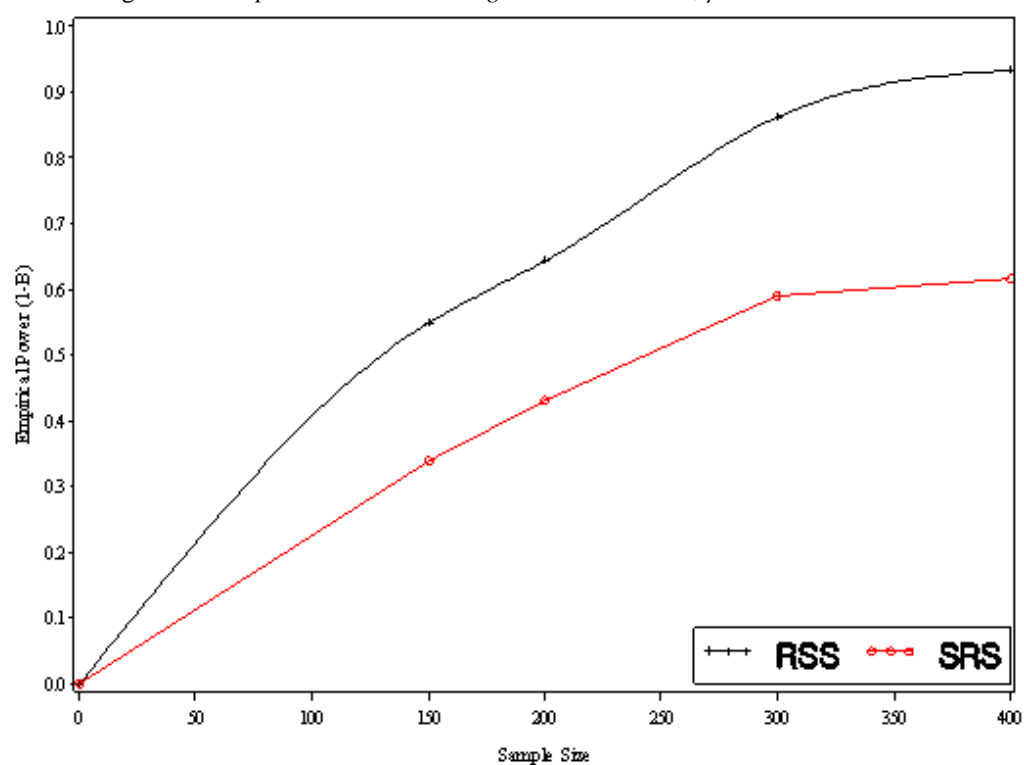
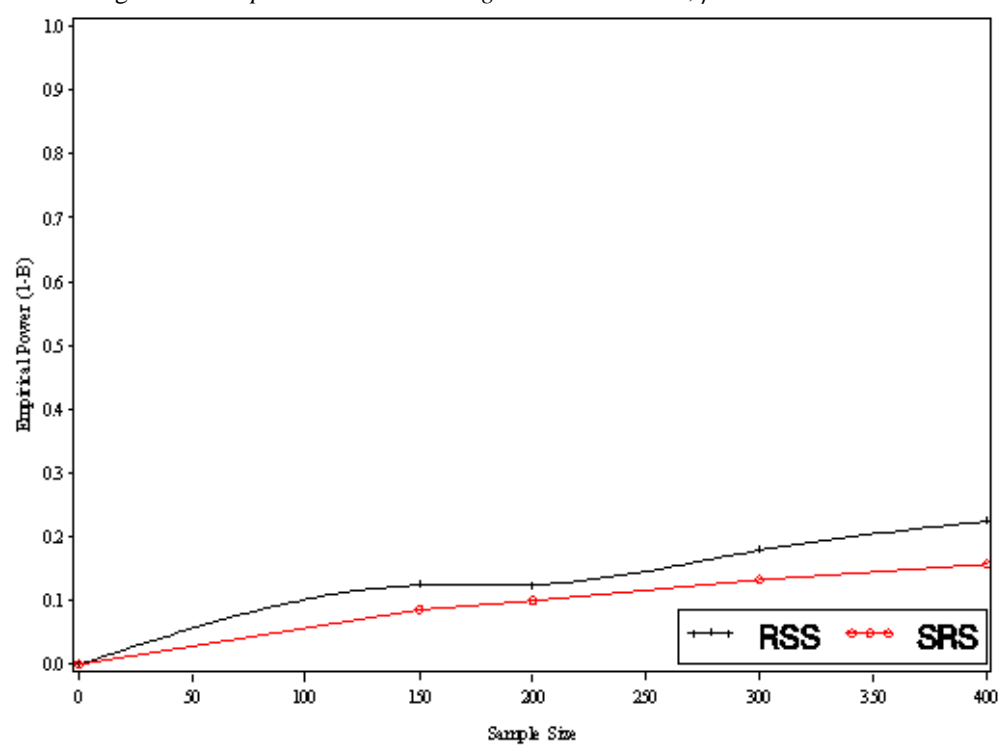
Figure 2.c. Empirical Power For Negative Correlations, $\rho = -0.3$ and $d = 0.8$.Figure 2.d. Empirical Power For Negative Correlations, $\rho = -0.5$ and $d = 0.3$.

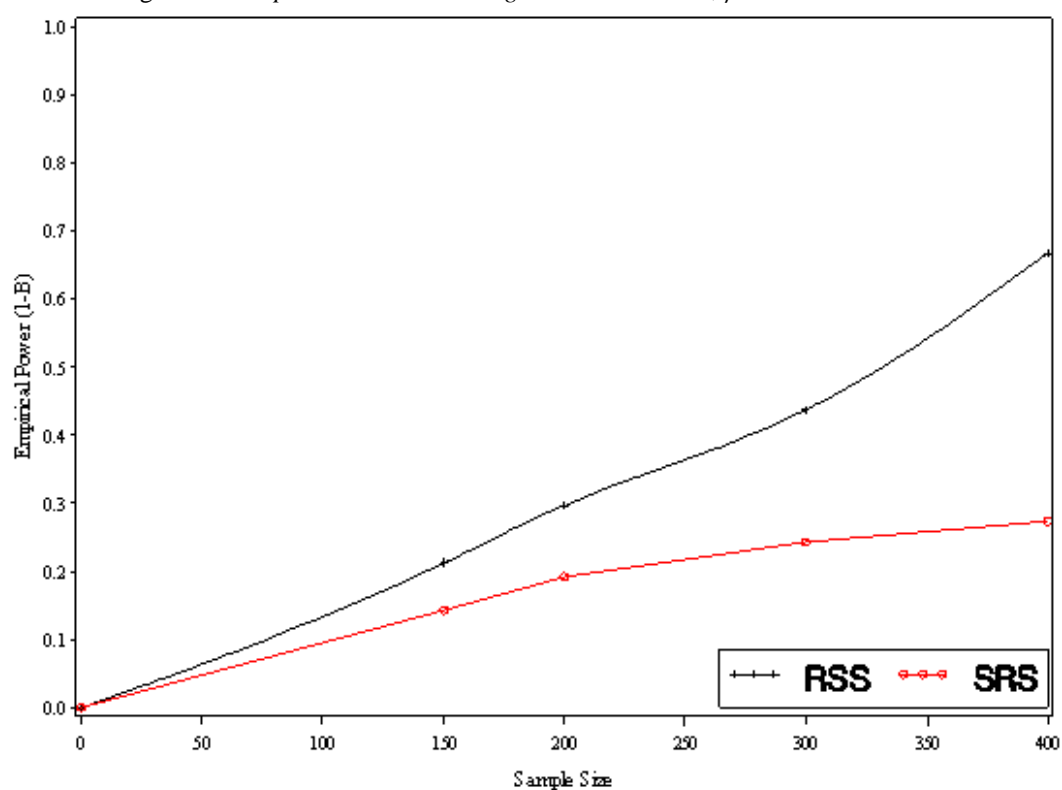
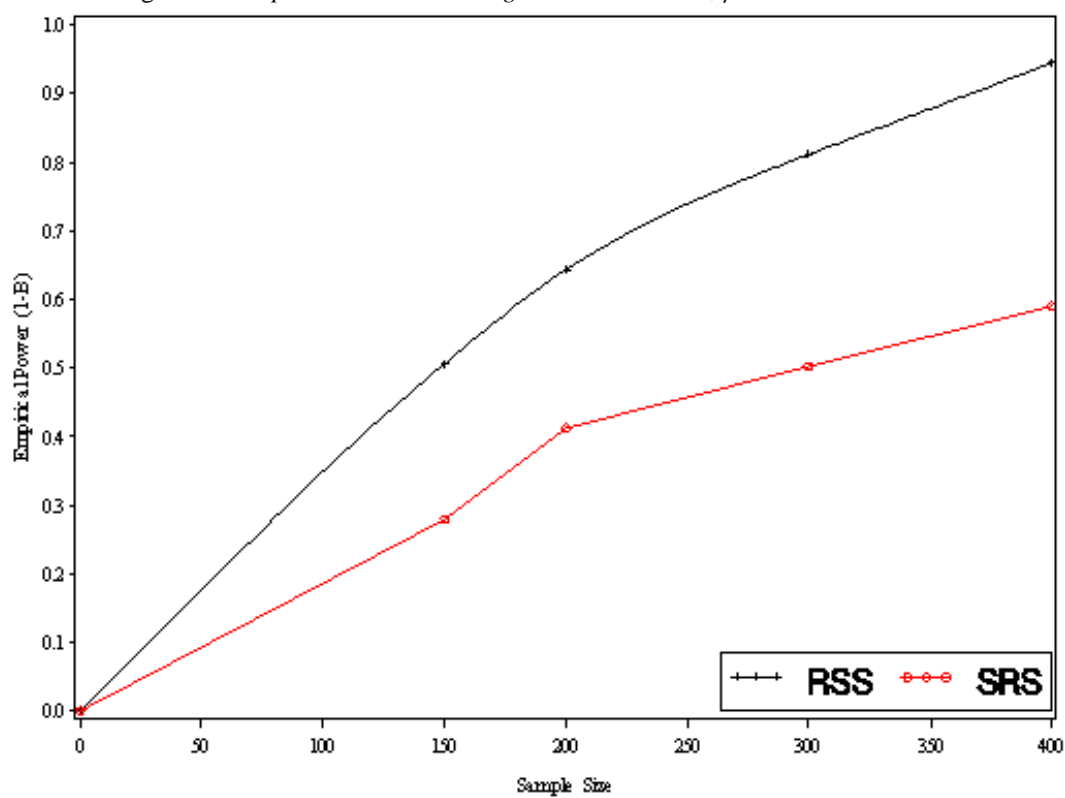
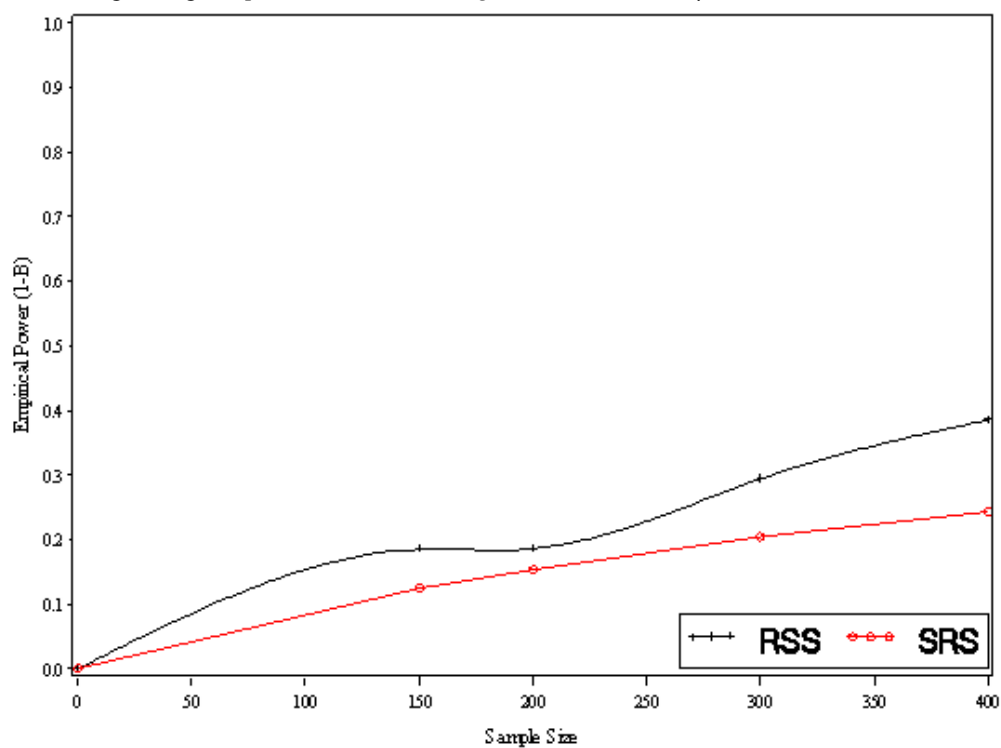
Figure 2.e. Empirical Power For Negative Correlations, $\rho = -0.5$ and $d = 0.5$.Figure 2.f. Empirical Power For Negative Correlations, $\rho = -0.5$ and $d = 0.8$.

Figure 2.g. Empirical Power For Negative Correlations, $\rho = -0.9$ and $d = 0.5$.Figure 2.h. Empirical Power For Negative Correlations, $\rho = -0.9$ and $d = 0.8$.